THE MATHEMATICAL MODELLING OF SPORT: using recurrence formulas in Excel

By
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Table of Contents

About the Author

Tristan Barnett is a mathematician with a PhD in tennis statistics who has made a career out of predicting sporting outcomes. This includes setting prices prior and during a match in progress for international bookmakers Ladbrokes and Centrebet, and sports IT company Infoplum. He is most recognized for his tennis predictions with appearances on SEN sports radio, 3RRR 'Run Like You Stole Something' sports segment and several articles in the Australian Financial Review. He is also a consultant for Tennis Australia in performance analysis and has been a tutor/lecturer for a 'Chance and Gaming' subject at Swinburne University. Tristan has self-published books on ‘The Mathematics of Tennis’, ‘Operations Research in Tennis’ and ‘Resolving Problem Gambling: a mathematical approach’. Tristan founded Strategic Games which provides information on the mathematics in sport, gambling and conflicts (html).

Preface

In 2012 the author self-published a book 'The Mathematics of Tennis' (pdf), which outlines recurrence formulas for the chances of winning and the parameters of distribution for the number of points remaining in a match from any score line within the match for either player A of player B serving. It also outlines recurrence formulas to enable the distribution of points within a deuce or tiebreak game, the distribution of games within a tiebreak or advantage set, the distribution of sets within a best-of-3 or best-of-5 set match, and also uses the Normal Power approximation formula to estimate the distribution of points remaining in a match from any score line within a match. The boundary conditions for backwards recurrence formulas and the initial conditions for forward recurrence formulas are also given. It can then be argued that the complete set of mathematical modelling has been obtained for tennis. Using the approach of recurrence formulas enables one to easily implement in spreadsheets such as Excel to obtain numerical results without the need for programming. An Excel spreadsheet calculator where the Normal Power approximation is applied to estimate the distribution of points remaining in the set from any score line within a set can be freely downloaded from (xlsx). Another Excel spreadsheet calculator where simulation is applied to estimate the distribution of points remaining in the match (for improved accuracy to the Normal Power Approximation) can be freely downloaded from (xlsx). By estimating the amount of time to play a point, this latter calculator also provides the parameters of distribution for the time duration in a match and the distribution of time remaining in the match from any score line within the match. Using prediction modelling techniques as outlined in the final chapter in 'The Mathematics of Tennis', this tennis model has been used successfully in industry for bookmakers' Ladbrokes and Centrebet and sports IT company Infoplum.

Chapter 1 outlines all the different types of tennis scoring systems used for men's and women's singles, doubles and mixed doubles matches on the main tour (a total of 8). Using four scoring systems, recurrence formulas are given for the chances of winning a match for any score line in the match, mean and variance of points remaining in a game for any point score in the game, mean and variance of games remaining in a set for any point and game score in the set, and the mean and variance of sets remaining in a match from any point, game and set score in the match. Similarly, chapter 2 gives recurrence formulas in badminton for the chances of winning a match from any score line within the match, mean and variance of points remaining in a game from any point score within the game, and the mean and variance of games remaining in a match from any point and game score within the match. Although these formulas can be readily obtained from tennis, the author had been in collaboration with RMIT University who were assisting Badminton Australia at the time and therefore felt it was necessary to provide these formulas directly to assist in performance analysis. Chapter 3 gives recurrence formulas in volleyball for the chances of winning a match from any score line within the match, mean and variance of points remaining in a set from any point score within the set, and the mean and variance of sets remaining in a match from any point and set score within the match. Volleyball is more complicated to analyze than tennis or badminton due to the rotation on serve and therefore there is still work to be done in the distribution of points in a set, distribution of sets in a match, first four moments of the number of points remaining in the set, the first four moments of the number of sets remaining in the match, and ultimately the first four moments of the number of points remaining in the match from any point and set score within the match. Chapter 3 outlines the different scoring systems across badminton, table tennis, volleyball, beach volleyball and squash. The mathematics of table tennis and squash can readily be obtained from tennis and a spreadsheet calculator based on the recurrence formulas given in volleyball are given for badminton, table tennis, volleyball, beach volleyball and squash (xlsx).

Chapter 4 gives the distribution of margins and the mean and variance of margins are obtained for round matches in Australian Rules Football (AFL) using recursive formulas. The probabilities of winning for Team A and Team B can be obtained from the distribution of margins. The probabilities of winning finals and grand finals matches are then obtained. The distribution of the total number of points for team A, the distribution of the total number of points for team B, and the distribution of the total number of points for both teams combined can also be obtained for round matches using recursive formulas similar to the distribution of margins. These formulas are documented in (pdf). Chapter 4 also provides formulas to obtain the mean and variance of the number or margins. The coefficients of skewness and kurtosis for the number of margins can readily be obtained, and thus the parameters of distribution for the number of margins, total number of points for team A, the total number of points for team B, and the total number of points for both teams combined can also be obtained for round matches. Chapter 4 outlines the different scoring systems used in AFL, rugby union, rugby league and soccer, and calculators have been developed in AFL (xlsx), rugby union (xlsx), rugby league (xlsx) and soccer (xlsx). The analysis for these sports is similar to AFL and in fact the same type of modelling can be used in any head-to-head sport with a fixed duration of time. The author successfully applied these mathematical models whilst working for bookmaker Centrebet in AFL, basketball, ice hockey, soccer and American football (NFL). The author successfully applied these mathematical models whilst working for Infoplum in AFL, rugby league, rugby union and soccer. Also, the author successfully developed simulation models for all three forms of crickets whilst working for Infoplum; being One-Day international cricket, T20 cricket and test cricket.

Chapter 1: Tennis

1. Scoring Systems

To win a deuce game requires winning four points. However, if the scores are level after six points have been played (known as deuce) then play continues indefinitely until a player has established a two-point lead and wins the game.

To win a no-ad game requires winning four points. However, if the scores are level after six points have been played then the player that wins the next point wins the game.

To win a first-to-7 points tiebreak game requires winning seven points. However, if the scores are level at 6-points-all then play continues indefinitely until a player has established a two-point lead and wins the game. The player that received last in the preceding game serves the first point, and then the players alternate serving every two points.

To win a first-to-10 points tiebreak game requires winning ten points. However, if the scores are level at 9 points-all then play continues indefinitely until a player has established a two-point lead and wins the game. The player that received last in the preceding game serves the first point, and then the players alternate serving every two points.

To win a tiebreak set requires winning six games and be ahead by at least two games. Players alternate service each game. However, if the scores are level at 6 games-all then a tiebreak game is played to decide the set.

To win an advantage set with requires winning six games. However, if the scores are level at 5 games-all then the set continues indefinitely until one player is two games ahead, and wins the set. Players alternate service each game.

The scoring structure for an all tiebreak set match is defined as follows. For a best-of-3 all tiebreak set match, the first player to reach 2 tiebreak sets wins the match. For a best-of-5 all tiebreak set match, the first player to reach 3 tiebreak sets wins the match. Usually the toss of a coin decides who will be serving the first game of the match. The server for the first game in the other sets will be the player who was receiving the last game in the prior set. If a set finishes with a tiebreak game, then the player that served first in that set, will be receiving for the first game in the next set.

The scoring structure for a final set advantage match is defined as follows. For a best-of-5 final set advantage match, the first player to reach 3 sets wins the match. The first 4 sets are tiebreak sets and the 5th set is played as an advantage set. For a best-of-3 final set advantage match, the first player to reach 2 sets wins the match. The first 2 sets are tiebreak sets and the 3rd set is played as an advantage set. Usually the toss of a coin decides who will be serving the first game of the match. The server for the first game in the other sets will be the player who was receiving the last game in the prior set. If a set finishes with a tiebreak game, then the player that served first in that set, will be receiving for the first game in the next set.

The scoring structure of a final set ending with a tiebreak game is the same as a first-to-10 points tiebreak game.

Table 1 lists the type of scoring systems used for men's and women's singles, doubles and mixed doubles matches on the main tour. All early sets are tiebreak sets with a first-to-7 points tiebreak games played at 6-games all in the set.

System	Event	Games	Final Set	Match
1	US Open women's singles Aust./French/US Open women's doubles Men's and women's singles	Deuce	Tiebreak First-to-7 points tiebreak game	3 sets
2	Australian Open women's singles	Deuce	Tiebreak First-to 10 points tiebreak game	3 sets
3	French/Wimbledon women's singles Olympics men's and women's singles Olympics men's and women's doubles Wimbledon women's doubles Wimbledon mixed doubles	Deuce	Advantage	3 sets
4	Aust/French/US Open mixed doubles	Deuce	First-to-10 points tiebreak game	3 sets
5	Men's and women's doubles	No-ad	First-to-10 points tiebreak game	3 sets
6	US Open men's singles Aust./French/US Open men's doubles	Deuce	Tiebreak First-to-7 points tiebreak game	5 sets
7	Australian Open men's singles	Deuce	Tiebreak First-to 10 points tiebreak game	5 sets
8	French/Wimbledon men's singles Olympics men's singles (Gold medal) Olympics men's doubles (Gold medal) Wimbledon men's doubles	Deuce	Advantage	5 sets

Table 1: Types of scoring systems for men's and women's singles, doubles and mixed doubles matches on the main tour

2. Chances of Winning

Only formulas for systems 1,3,6 and 8 will be covered in this book. The remaining systems can readily be obtained by only a slight change is the recursion formulas and boundary conditions.

2.1 Notation for winning a game

Let $P_A^{pg}(a,b)$ and $P_B^{pg}(a,b)$ represent the probabilities that player A wins the deuce game from the point score is (a,b) given player A and player B are serving respectively, where a is the point score for player A and b is the point score for player B.

Let $P_A^{pgT}(a,b)$ and $P_B^{pgT}(a,b)$ represent the probabilities of player A winning a first-to-7 points tiebreak game from point score (a,b) given player A and player B are serving respectively, where a is the point score for player A and b is the point score for player B.

Let p_A represent a constant probability of player A winning a point on serve
Let q_A represent a constant probability of player A losing a point on serve
Let p_B represent a constant probability of player B winning a point on serve
Let q_B represent a constant probability of player B losing a point on serve

It follows that $q_A = 1-p_A$ and $q_B = 1-p_B$

Let p_A^g and p_B^g represent the probabilities of player A and player B winning a game on serve from the outset.

Let q_A^g and q_B^g represent the probabilities of player A and player B losing a game on serve from the outset.

It follows that $p_A^g=P_A^{pg}(0,0)$, $p_B^g=1-P_B^{pg}(0,0)$, $q_A^g=1-p_A^g$ and $q_B^g = 1-p_B^g$.

Let p_A^{gT} and p_B^{gT} represent the probabilities of player A and player B respectively winning a tiebreak game on serve from the outset.

It follows that $p_A^{gT}=P_A^{pgT}(0,0)$ and $p_B^{gT}= 1-P_B^{pgT}(0,0)$.

Since there is no advantage in serving first in a tiebreak game, that is $p_A^{gT}=1-p_B^{gT}$, it becomes convenient to let p^{gT} and $q^{gT}=1-p^{gT}$ represent the respective probabilities of player A and player B winning a tiebreak game from the outset.

2.2 Winning a game

Recurrence Formula
$P_A^{pg}(a,b) = p_A P_A^{pg}(a+1,b)+q_A P_A^{pg}(a,b+1)$

Boundary Values
$P_A^{pg}(a,b) = 1$, if a=4 and $b \leq 2$
$P_A^{pg}(a,b) = 0$, if b=4 and $a \leq 2$
$P_A^{pg}(3,3) = p_A^2/(p_A^2+q_A^2)$

Table 1 represents the conditional probabilities of player A winning the game from various score lines for $p_A = 0.6$. It indicates that a player with a 60% chance of winning a point has a 73.6% chance of winning the game. Note that since advantage server is logically equivalent to 40-30, and advantage receiver is logically equivalent to 30-40, the required statistics can be found from these cells. Also, worth noting is that the chances of winning from deuce and 30-30 are the same.

				B score		
		0	15	30	40	game
	0	0.736	0.576	0.369	0.150	0
	15	0.842	0.714	0.515	0.249	0
A score	30	0.927	0.847	0.692	0.415	0
	40	0.980	0.951	0.877	0.692	
	game	1	1	1		

Table 1: The conditional probabilities of player A winning the game on serve from various score lines for p_A=0.6

Recurrence Formulas

$P_A^{pgT}(a,b) = p_A P_B^{pgT}(a + 1,b) + q_A P_B^{pgT}(a,b + 1)$, if (a + b) is even

$P_A^{pgT}(a,b) = p_A P_A^{pgT}(a + 1,b) + q_A P_A^{pgT}(a,b + 1)$, if (a + b) is odd

Boundary Values

$P_A^{pgT}(a,b) = 1$, if $a = 7$ and $0 \le b \le 5$

$P_A^{pgT}(a,b) = 0$, if $b = 7$ and $0 \le a \le 5$

$P_A^{pgT}(6,6) = p_A q_B/(p_A q_B+q_A p_B)$

Table 2 represents the conditional probabilities of player A winning the tiebreak game on serve from various score lines for $p_A = 0.62$ and $p_B = 0.60$. Table 3 is represented similarly, with player A winning a tiebreak game from various score lines given player B is serving. It indicates that player A has a 0.533 probability of winning the tiebreak game from the outset for player A or player B serving. Note how the calculations are obtained by the interconnection of both sheets. For example

$P_A^{pgT}(0,0)=p_A P_B^{pgT}(1,0)+q_A P_B^{pgT}(0,1) = 0.62*0.620+0.38*0.389 = 0.533$.

					B score				
		0	1	2	3	4	5	6	7
	0	0.533	0.441	0.295	0.205	0.096	0.044	0.008	0
	1	0.670	0.530	0.431	0.271	0.174	0.066	0.020	0
	2	0.755	0.680	0.528	0.417	0.240	0.134	0.032	0
A score	3	0.868	0.773	0.695	0.526	0.399	0.197	0.080	0
	4	0.926	0.892	0.798	0.716	0.523	0.372	0.129	0
	5	0.977	0.949	0.921	0.834	0.750	0.521	0.323	0
	6	0.994	0.991	0.975	0.959	0.891	0.818	0.521	
	7	1	1	1	1	1	1		

Table 2: The conditional probabilities of player A winning the tiebreak game on serve from various score lines for $p_A = 0.62$ and $p_B = 0.60$

					B score				
		0	1	2	3	4	5	6	7
	0	0.533	0.389	0.295	0.166	0.096	0.031	0.008	0
	1	0.620	0.530	0.374	0.271	0.135	0.066	0.013	0
	2	0.755	0.626	0.528	0.355	0.240	0.098	0.032	0
A score	3	0.834	0.773	0.635	0.526	0.328	0.197	0.052	0
	4	0.926	0.858	0.798	0.647	0.523	0.286	0.129	0
	5	0.967	0.949	0.890	0.834	0.669	0.521	0.208	0
	6	0.994	0.985	0.975	0.934	0.891	0.713	0.521	
	7	1	1	1	1	1	1		

Table 3: The conditional probabilities of player A winning the tiebreak game on serve from various score lines given player B is serving for $p_A = 0.62$ and $p_B = 0.60$

The formulas for $P_B^{pgT}(a,b)$ can be obtained similarly.

2.3 Notation for winning a set

Let $P_A^{gsT}(c,d)$ and $P_B^{gsT}(c,d)$ represent the probabilities of player A winning a tiebreak set with deuce games played prior to 6-games all and a first-to-7 points tiebreak game played at 6-games all from game score (c,d) given player A and player B are serving respectively, where c is the game score for player A and d is the game score for player B.

Let $P_A^{gs}(c,d)$ and $P_B^{gs}(c,d)$ represent the probabilities of player A winning an advantage set with deuce games played from game score (c,d) given player A and player B are serving respectively, where c is the game score for player A and d is the game score for player B.

Let $P_A^{psT}(a,b : c,d)$ represent the probability of player A winning a tiebreak set on serve with deuce games played prior to 6-games all and a first-to-7 points tiebreak game played at 6-games all in the set from (c,d) in games and (a,b) in points.

Let $P_A^{ps}(a,b : c,d)$ represent the probability of player A winning an advantage set on serve with deuce games played from (c,d) in games and (a,b) in points.

Let p_A^{sT} and p_B^{sT} represent the probabilities of player A and player B respectively winning a tiebreak set on serve from the outset.

It follows that $p_A^{sT} = P_A^{gsT}(0,0)$ and $p_B^{sT} = 1 - P_B^{gsT}(0,0)$.

Since there is no advantage in serving first in a tiebreak set. That is: $p_A^{sT} = 1 - p_B^{sT}$, it becomes convenient to let p^{sT} and $q^{sT} = 1 - p^{sT}$ represent the respective probabilities of player A and player B winning a tiebreak set from the outset.

Let p_A^{s} and p_B^{s} represent the probabilities of player A and player B respectively winning an advantage set on serve from the outset.

It follows that $p_A^{s} = P_A^{gs}(0,0)$ and $p_B^{s} = 1 - P_B^{gs}(0,0)$.

Since there is no advantage in serving first in an advantage set. That is: $p_A^{s} = 1 - p_B^{s}$, it becomes convenient to let p^{s} and $q^{s} = 1 - p^{s}$ represent the respective probabilities of player A and player B winning an advantage set from the outset.

2.4 Winning a set

Recurrence Formula

$P_A^{gsT}(c,d) = p_A^{g} P_B^{gsT}(c + 1,d) + q_A^{g} P_B^{gsT}(c,d + 1)$

Boundary Values

$P_A^{gsT}(c,d) = 1$, if $c = 6$ and $0 \leq d \leq 4$; (7,5)

$P_A^{gsT}(c,d) = 0$, if $d = 6$ and $0 \leq c \leq 4$; (5,7)

$P_A^{gsT}(6,6) = p^{gT}$

$P_A^{psT}(a,b : c,d) = P_A^{pg}(a,b) P_B^{gsT}(c + 1,d) + (1 - P_A^{pg}(a,b)) P_B^{gsT}(c,d + 1)$, if $(c,d) \neq (6,6)$

$P_A^{psT}(a,b : c,d) = P_A^{pgT}(a,b)$, if $(c,d) = (6,6)$

Tables 4 and 5 show the probabilities of player A winning the tiebreak set, given $p_A = 0.62$ and $p_B = 0.60$. It indicates that player A has a 0.568 probability of winning the set from the outset for player A or player B serving. Notice how the cells $p_A^{g} = P_A^{pg}(0,0)$ and $q_A^{g} = 1 - P_A^{pg}(0,0)$, which represent the probability of player A winning and losing a game when serving respectively, is used in the recurrence formula for a tiebreak set.

					B score				
		0	1	2	3	4	5	6	7
	0	0.568	0.497	0.269	0.191	0.052	0.018	0	
	1	0.766	0.563	0.484	0.231	0.146	0.023	0	
	2	0.824	0.781	0.557	0.469	0.182	0.086	0	
A score	3	0.944	0.846	0.804	0.552	0.448	0.111	0	
	4	0.972	0.963	0.877	0.839	0.546	0.419	0	
	5	0.977	0.988	0.983	0.924	0.897	0.541	0.413	0
	6	1	1	1	1	1	0.895	0.533	
	7						1		

Table 4: The conditional probabilities of player A winning the tiebreak set on serve from various score lines for $p_A = 0.62$ and $p_B = 0.60$

					B score				
		0	1	2	3	4	5	6	7
	0	0.568	0.346	0.269	0.099	0.052	0.006	0	
	1	0.632	0.563	0.317	0.231	0.065	0.023	0	
	2	0.824	0.634	0.557	0.279	0.182	0.029	0	
A score	3	0.879	0.846	0.638	0.552	0.226	0.111	0	
	4	0.972	0.906	0.877	0.646	0.546	0.143	0	
	5	0.991	0.988	0.944	0.924	0.662	0.541	0.141	0
	6	1	1	1	1	1	0.656	0.533	
	7						1		

Table 5: The conditional probabilities of player A winning the tiebreak set from various score lines for $p_A = 0.62$ and $p_B = 0.60$ given player B is serving

Recurrence Formula

$P_A^{gs}(c,d) = p_A^g P_B^{gs}(c + 1,d) + q_A^g P_B^{gs}(c,d + 1)$

Boundary Values

$P_A^{gs}(c,d) = 1$, if $c = 6$ and $0 \le d \le 4$

$P_A^{gs}(c,d) = 0$, if $d = 6$ and $0 \le c \le 4$

$P_A^{gs}(5,5) = p_A^g q_B^g/(p_A^g\, q_B^g + q_A^g\, p_B^g)$

$P_A^{ps}(a,b : c,d) = P_A^{pg}(a,b) P_B^{gs}(c + 1,d) + (1 - P_A^{pg}(a,b)) P_B^{gs}(c,d+1)$

Tables 6 and 7 show the conditional probabilities of player A winning the advantage set, given $p_A = 0.62$ and $p_B = 0.60$. It indicates that player A has a 0.572 probability of winning the set from the outset for player A or player B serving.

					B score			
		0	1	2	3	4	5	6
	0	0.572	0.501	0.272	0.194	0.053	0.018	0
	1	0.769	0.567	0.489	0.235	0.149	0.023	0
	2	0.827	0.785	0.563	0.474	0.185	0.088	0
A score	3	0.945	0.849	0.808	0.558	0.455	0.114	0
	4	0.973	0.964	0.880	0.843	0.554	0.430	0
	5	0.997	0.988	0.984	0.926	0.900	0.554	
	6	1	1	1	1	1		

Table 6: The conditional probabilities of player A winning the advantage set from various score lines for $p_A = 0.62$ and $p_B = 0.60$, and player A serving

					B score			
		0	1	2	3	4	5	6
	0	0.572	0.350	0.272	0.101	0.053	0.006	0
	1	0.636	0.567	0.322	0.235	0.066	0.023	0
	2	0.827	0.638	0.563	0.284	0.185	0.030	0
A score	3	0.882	0.849	0.643	0.558	0.230	0.114	0
	4	0.973	0.909	0.880	0.653	0.554	0.146	0
	5	0.991	0.988	0.946	0.926	0.672	0.554	
	6	1	1	1	1	1		

Table 7: The conditional probabilities of player A winning the advantage set from various score lines for $p_A = 0.62$ and $p_B = 0.60$, and player B serving

Similar formulas can be obtained for $P_B^{psT}(a,b : c,d)$ and $P_B^{ps}(a,b : c,d)$.

2.5 Notation for winning a match

Let $P^{sm3T}(e,f)$ represent the probabilities of player A winning a best-of-3 set match with deuce games played prior to 6-games all in each set and a first-to-7 points tiebreak game played at 6-games all in each set from set score (e,f), where e is the set score for player A and f is the set score for player B.

Let $P^{sm3}(e,f)$ represent the probabilities of player A winning a best-of-3 set match with deuce games played prior to 6-games all in each set, a first-to-7 points tiebreak game played at 6-games all in early sets and an advantage set played in the final set from set score (e,f), where e is the set score for player A and f is the set score for player B.

Let $P^{sm5T}(e,f)$ represent the probabilities of player A winning a best-of-5 set match with deuce games played prior to 6-games all in each set and a first-to-7 points tiebreak game played at 6-games all in each set from set score (e,f), where e is the set score for player A and f is the set score for player B.

Let $P^{sm5}(e,f)$ represent the probabilities of player A winning a best-of-5 set match with deuce games played prior to 6-games all in each set, a first-to-7 points tiebreak game played at 6-

games all in early sets and an advantage set played in the final set from set score (e,f), where e is the set score for player A and f is the set score for player B.

Let $P_A^{pm3T}(a,b : c,d : e,f)$ represent the probability of player A winning a best-of-3 set match on serve with deuce games played prior to 6-games all in each set and a first-to-7 points tiebreak game played at 6-games all in all sets from (e,f) in sets, (c,d) in games and (a,b) in points.

Let $P_A^{pm3}(a,b : c,d : e,f)$ represent the probability of player A winning a best-of-3 set match on serve with deuce games played prior to 6-games all in each set, a first-to-7 points tiebreak game played at 6-games all in early sets and an advantage set played in the final set from (e,f) in sets, (c,d) in games and (a,b) in points.

Let $P_A^{pm5T}(a,b : c,d : e,f)$ represent the probability of player A winning a best-of-5 set match on serve with deuce games played prior to 6-games all in each set and a first-to-7 points tiebreak game played at 6-games all in each set from (e,f) in sets, (c,d) in games and (a,b) in points.

Let $P_A^{pm5}(a,b : c,d : e,f)$ represent the probability of player A winning a best-of-5 set match on serve with deuce games played prior to 6-games all in each set, a first-to-7 points tiebreak game played at 6-games all in early sets and an advantage set played in the final set from (e,f) in sets, (c,d) in games and (a,b) in points.

2.6 Winning a match

System 1

Recurrence Formula

$P^{sm3T}(e,f) = p^{sT}P^{sm3T}(e + 1,f) + q^{sT}P^{sm3T}(e,f + 1)$

Boundary Values

$P^{sm3T}(e,f) = 1$, if $e = 2$ and $f \leq 1$

$P^{sm3T}(e,f) = 0$, if $f = 2$ and $e \leq 1$

$P_A^{pm3T}(a,b : c,d : e,f) = P_A^{psT}(a,b : c,d)\ P^{sm3T}(e+1,f)+(1-P_A^{psT}(a,b : c,d))\ P^{sm3T}(e,f+1)$

System 3

Recurrence Formula

$P^{sm3}(e,f) = p^{sT}\ P^{sm3}(e + 1,f) + q^{sT}\ P^{sm3}(e,f + 1)$

Boundary Values

$P^{sm3}(e,f) = 1$, if $e = 2$ and $f = 0$

$P^{sm3}(e,f) = 0$, if $f = 2$ and $e = 0$

$P^{sm3}(1,1) = p^{s}$

$P_A^{pm3}(a,b : c,d : e,f)$
$= P_A^{psT}(a,b : c,d)\ P^{sm3}(e+1,f) + (1-P_A^{psT}(a,b : c,d))\ P^{sm3}(e,f+1)$, if $(e,f) \neq (1,1)$
$P_A^{pm3}(a,b : c,d : e,f) = P_A^{ps}(a,b : c,d)$, if $(e,f) = (1,1)$

System 6

Recurrence Formula

$P^{sm5T}(e,f) = p^{sT}\ P^{sm5T}(e+1,f) + q^{sT}\ P^{sm5T}(e,f+1)$

Boundary Values

$P^{sm5T}(e,f) = 1$, if $e = 3$ and $f \leq 2$
$P^{sm5T}(e,f) = 0$, if $f = 3$ and $e \leq 2$

$P_A^{pm5T}(a,b : c,d : e,f) = P_A^{psT}(a,b : c,d)\ P^{sm5T}(e+1,f) + (1-P_A^{psT}(a,b : c,d))\ P^{sm5T}(e,f+1)$

Table 8 shows the conditional probabilities of player A winning a best-of-5 all tiebreak set match (System 6), given $p_A = 0.62$ and $p_B = 0.60$. It indicates that player A has a 0.626 probability of winning the match from the outset. It also shows that a small increase on serve for the stronger player magnifies throughout the match. When $p_A = 0.62$ and $p_B = 0.60$, this 0.02 increase in probability on serve for player A, magnifies to a 0.07 increase in probability to win a set, and a 0.13 increase in probability to win the match.

			B score		
		0	1	2	3
	0	0.626	0.421	0.183	0
A score	1	0.782	0.601	0.323	0
	2	0.919	0.813	0.568	0
	3	1	1	1	

Table 8: The conditional probabilities of player A winning a best-of-5 all tiebreak set match from various score lines for $p_A = 0.62$ and $p_B = 0.60$

System 8

Recurrence Formula

$P^{sm5}(e,f) = p^{sT}\ P^{sm5}(e+1,f) + q^{sT}\ P^{sm5}(e,f+1)$

Boundary Values

$P^{sm5}(e,f) = 1$, if $e = 3$ and $f \leq 1$
$P^{sm5}(e,f) = 0$, if $f = 3$ and $e \leq 1$
$P^{sm5}(2,2) = p^{s}$

$P_A^{pm5}(a,b : c,d : e,f) = P_A^{psT}(a,b : c,d)\ P^{sm5}(e+1,f)+(1-P_A^{psT}(a,b : c,d))P^{sm5}(e,f+1)$, if $(e,f) \neq (2,2)$
$P_A^{pm5}(a,b : c,d : e,f) = P_A^{ps}(a,b : c,d)$, if $(e,f) = (2,2)$

Table 9 shows the conditional probabilities of player A winning a best-of-5 final set advantage match (System 8), given p_A = 0.62 and p_B = 0.60. It indicates that player A has a 0.627 probability of winning the match from the outset.

			B score		
		0	1	2	3
	0	0.627	0.422	0.184	0
A score	1	0.783	0.603	0.325	0
	2	0.920	0.815	0.572	
	3	1	1		

Table 9: The conditional probabilities of player A winning a best-of-5 final set advantage match from various score lines for p_A = 0.62 and p_B = 0.60

3. Match Duration

3.1 Notation for points remaining in a game

Let $Y_A^{pg}(a,b)$ and $Y_B^{pg}(a,b)$ be random variables of the number of points remaining in a deuce game at point score (a,b) for player A and player B serving respectively.

Let $Y_A^{pgT}(a,b)$ and $Y_B^{pgT}(a,b)$ be random variables of the number of points remaining in a first-to 7 tiebreak game at point score (a,b) for player A and player B serving respectively.

Let $\mu(Y_A^{pg}(a,b))$ and $\mu(Y_B^{pg}(a,b))$ represent the mean number of points remaining in a deuce game at point score (a,b) for player A and player B serving respectively.

Let $\mu(Y_A^{pgT}(a,b))$ and $\mu(Y_B^{pgT}(a,b))$ represent the mean number of points remaining in a first-to-7 tiebreak game at point score (a,b) for player A and player B serving respectively.

Let $\sigma^2(Y_A^{pg}(a,b))$ and $\sigma^2(Y_B^{pg}(a,b))$ represent the variance of the number of points remaining in a deuce game at point score (a,b) for player A and player B serving respectively.

Let $\sigma^2(Y_A^{pgT}(a,b))$ and $\sigma^2(Y_B^{pgT}(a,b))$ represent the variance of the number of points remaining in a first-to-7 tiebreak game at point score (a,b) for player A and player B serving respectively.

3.2 Mean and variance of points remaining in a game

Recurrence Formula

$\mu(Y_A^{pg}(a,b)) = 1 + p_A\mu(Y_A^{pg}(a+1,b)) + q_A\mu(Y_A^{pg}(a,b+1))$

Boundary Values

$\mu(Y_A^{pg}(a,b)) = 0$, if $b = 4$ and $a \leq 2$; $a = 4$ and $b \leq 2$

$\mu(Y_A^{pg}(3,3)) = 2/ (p_A^2+q_A^2)$

Table 10 lists the mean number of points remaining in a game from point score (a,b) for player A serving with $p_A = 0.6$. It indicates that the mean number of points remaining in such a game is 6.5.

				B score		
		0	15	30	40	game
	0	6.5	6.0	4.8	2.8	0
	15	5.2	5.0	4.5	3.0	0
A score	30	3.6	3.7	3.8	3.3	0
	40	1.8	2.0	2.5	3.8	
	game	0	0	0		

Table 10: The mean number of points remaining in a game from various score lines for player A serving with $p_A = 0.6$

Recurrence Formula

$\sigma^2(Y_A^{pg}(a,b))$

$= p_A\sigma^2(Y_A^{pg}(a + 1,b)) + q_A\sigma^2(Y_A^{pg}(a,b + 1)) + p_Aq_A(\mu(Y_A^{pg}(a + 1,b)) - \mu(Y_A^{pg}(a,b + 1)))^2$

Boundary Values

$\sigma^2(Y_A^{pg}(a,b)) = 0$, if $b = 4$ and $a \leq 2$; $a = 4$ and $b \leq 2$

$\sigma^2(Y_A^{pg}(3,3)) = 8p_Aq_A/(p_A^2+q_A^2)^2$

Table 11 lists the variance of the number of points remaining in a game from point score (a,b) for player A serving with $p_A = 0.6$. It indicates that the variance of the number of points remaining in such a game is 6.7.

				B score		
		0	15	30	40	game
	0	6.7	7.2	7.7	6.5	0
	15	6.2	6.7	7.4	7.3	0
A score	30	4.9	6.1	7.1	7.8	0
	40	2.6	4.1	6.4	7.1	
	Game	0	0	0		

Table 11: The variance of the number of points remaining in a game from various score lines for player A serving with $p_A = 0.6$

Recurrence Formulas

$\mu(Y_A^{pgT}(a,b)) = 1 + p_A\mu(Y_B^{pgT}(a + 1,b)) + q_A\mu(Y_B^{pgT}(a,b + 1))$, if $(a + b)$ is even

$\mu(Y_A^{pgT}(a,b)) = 1 + p_A\mu(Y_A^{pgT}(a + 1,b)) + q_A\mu(Y_A^{pgT}(a,b + 1))$, if $(a + b)$ is odd

Boundary Values

$\mu(Y_A^{pgT}(a,b)) = 0$, if $a = 7$ and $0 \leq b \leq 5$; $b=7$ and $0 \leq a \leq 5$

$\mu(Y_A^{pgT}(6,6)) = 2/(p_Aq_B+q_Ap_B)$

Table 12 represents the mean number of points remaining in a tiebreak game from various score lines with player A currently serving with $p_A = 0.62$ and $p_B = 0.60$. It indicates that the mean number of points remaining from the outset in such a game is 11.9.

					B score				
		0	1	2	3	4	5	6	7
	0	11.9	11.0	9.6	8.2	6.2	4.4	2.1	0
	1	10.7	10.1	9.2	7.9	6.4	4.2	2.4	0
	2	9.4	9.0	8.5	7.6	6.1	4.5	2.2	0
A score	3	7.6	7.7	7.3	6.9	6.0	4.3	2.7	0
	4	6.0	5.8	5.9	5.7	5.4	4.6	2.7	0
	5	3.7	4.1	3.9	4.3	4.0	4.2	3.6	0
	6	1.9	1.7	2.0	1.9	2.3	2.6	4.2	
	7	0	0	0	0	0	0		

Table 12: The mean number of points remaining in a tiebreak game from various score lines with player A currently serving with $p_A = 0.62$ and $p_B = 0.60$

Recurrence Formulas

$\sigma^2(Y_A^{pgT}(a,b))$

$= p_A\sigma^2(Y_B^{pgT}(a + 1,b)) + q_A\sigma^2(Y_B^{pgT}(a,b + 1)) + p_Aq_A(\mu(Y_B^{pgT}(a + 1,b)) - \mu(Y_B^{pgT}(a,b + 1)))^2$, if (a + b) is even

$\sigma^2(Y_A^{pgT}(a,b))$

$= p_A\sigma^2(YA^{pgT}(a + 1,b)) + q_A\sigma^2(Y_A^{pgT}(a,b + 1)) + p_Aq_A(\mu(Y_A^{pgT}(a + 1,b)) - \mu(Y_A^{pgT}(a,b + 1)))^2$, if (a + b) is odd

Boundary Values

$\sigma^2(Y_A^{pgT}(a,b)) = 0$, if $a = 7$ and $0 \le b \le 5$; $b = 7$ and $0 \le a \le 5$

$\sigma^2(Y_A^{pgT}(6,6)) = 4(p_Ap_B+q_Aq_B)/(p_Aq_B+q_Ap_B)^2$

Table 13 represents the variance of the number of points remaining in a tiebreak game from various score lines with player A currently serving with $p_A = 0.62$ and $p_B = 0.60$. It indicates that the variance of the number of points remaining from the outset in such a game is 9.2.

					B score				
		0	1	2	3	4	5	6	7
	0	9.2	9.1	9.4	9.2	8.0	5.9	2.4	0
	1	9.1	8.9	8.8	8.8	8.7	6.4	3.4	0
	2	9.1	8.9	8.7	8.7	8.6	7.4	3.6	0
A score	3	8.6	8.8	8.5	8.6	8.8	8.1	5.8	0
	4	7.3	7.3	8.1	8.3	8.7	8.8	6.6	0
	5	4.7	5.5	6.3	7.5	8.5	9.3	9.9	0
	6	2.4	2.2	3.5	3.8	6.6	7.7	9.3	
	7	0	0	0	0	0	0		

Table 13: The variance of the number of points remaining in a tiebreak game from various score lines with player A currently serving with $p_A = 0.62$ and $p_B = 0.60$

Similar formulas can be obtained for $\mu(Y_B^{pg}(a,b))$, $\mu(Y_B^{pgT}(a,b))$, $\sigma^2(Y_B^{pg}(a,b))$ and $\sigma^2(Y_B^{pgT}(a,b))$.

3.2 Notation for games remaining in a set

Let $Y_A^{gsT}(c,d)$ and $Y_B^{gsT}(c,d)$ represent random variables of the number of games remaining in a tiebreak set from game score (c,d) for player A and player B serving respectively.

Let $Y_A^{gs}(c,d)$ and $Y_B^{gs}(c,d)$ represent random variables of the number of games remaining in an advantage set from game score (c,d) for player A and player B serving respectively.

Let $Y_A^{gsT}(a,b : c,d)$ and $Y_B^{gsT}(a,b : c,d)$ represent random variables of the number of games remaining in a tiebreak set from point and game score (a,b : c,d) for player A and player B serving respectively.

Let $Y_A^{gs}(a,b : c,d)$ and $Y_B^{gs}(a,b : c,d)$ represent random variables of the number of games remaining in an advantage set from point and game score (a,b : c,d) for player A and player B serving respectively.

Let $\mu(Y_A^{gsT}(c,d))$ and $\mu(Y_B^{gsT}(c,d))$ represent the mean number of games remaining in a tiebreak set from game score (c,d) for player A and player B serving respectively.

Let $\mu(Y_A^{gs}(c,d))$ and $\mu(Y_B^{gs}(c,d))$ represent the mean number of games remaining in an advantage set from game score (c,d) for player A and player B serving respectively.

Let $\mu(Y_A^{gsT}(a,b : c,d))$ and $\mu(Y_B^{gsT}(a,b : c,d))$ represent the mean number of games remaining in a tiebreak set from point and game score (a,b : c,d) for player A and player B serving respectively.

Let $\mu(Y_A^{gs}(a,b : c,d))$ and $\mu(Y_B^{gs}(a,b : c,d))$ represent the mean number of games remaining in an advantage set from point and game score (a,b : c,d) for player A and player B serving respectively.

Let $\sigma^2(Y_A^{gsT}(c,d))$ and $\sigma^2(Y_B^{gsT}(c,d))$ represent the variance of the number of games remaining in a tiebreak set from game score (c,d) for player A and player B serving respectively.

Let $\sigma^2(Y_A^{gs}(c,d))$ and $\sigma^2(Y_B^{gs}(c,d))$ represent the variance of the number of games remaining in an advantage set from game score (c,d) for player A and player B serving respectively.

Let $\sigma^2(Y_A^{gsT}(a,b : c,d))$ and $\sigma^2(Y_B^{gsT}(a,b : c,d))$ represent the variance of the number of games remaining in a tiebreak set from point and game score (a,b : c,d) for player A and player B serving respectively.

Let $\sigma^2(Y_A^{gs}(a,b : c,d))$ and $\sigma^2(Y_B^{gs}(a,b : c,d))$ represent the variance of the number of games remaining in a tiebreak set from point and game score (a,b : c,d) for player A and player B serving respectively.

3.4 Mean and variance of games remaining in a set

Recurrence Formula

$\mu(Y_A^{gsT}(c,d)) = 1 + p_A^g\mu(Y_B^{gsT}(c + 1,d)) + q_A^g\mu(Y_B^{gsT}(c,d + 1))$

Boundary Values

$\mu(Y_A^{gsT}(c,d)) = 0$, if $c = 6$ and $0 \le d \le 4$; $d = 6$ and $0 \le c \le 4$; (7,5); (5,7)

$\mu(Y_A^{gsT}(6,6)) = 1$

$\mu(Y_A^{gsT}(a,b : c,d)) = 1 + P_A^{pg}(a,b)\, \mu(Y_B^{gsT}(c + 1,d)) + (1-P_A^{pg}(a,b))\mu(Y_B^{gsT}(c,d + 1))$, if $(c,d) \neq (6,6)$

$\mu(Y_A^{gsT}(a,b : c,d)) = 1$, if $(c,d) = (6,6)$

Table 14 represents the mean number of games remaining in a tiebreak set from various score lines with player A currently serving with $p_A = 0.62$ and $p_B = 0.60$. It indicates that the mean number of games remaining from the outset in such a set is 10.0.

					B score				
		0	1	2	3	4	5	6	7
	0	10.0	9.2	7.9	6.3	4.3	2.3	0	
	1	8.7	8.3	7.4	6.1	4.4	2.3	0	
	2	7.2	7.0	6.6	5.8	4.2	2.4	0	
A score	3	5.4	5.4	5.2	5.0	4.2	2.3	0	
	4	3.5	3.5	3.6	3.5	3.7	3.0	0	
	5	1.5	1.5	1.5	1.7	1.6	2.6	1.8	0
	6	0	0	0	0	0	1.2	1	
	7						0		

Table 14: The mean number of games remaining in a tiebreak set from various score lines with player A currently serving with $p_A = 0.62$ and $p_B = 0.60$

Recurrence Formula

$\sigma^2(Y_A^{gsT}(c,d))$

$= p_A^g\, \sigma^2(Y_B^{gsT}(c + 1,d)) + q_A^g\, \sigma^2(Y_B^{gsT}(c,d + 1)) + p_A^g\, q_A^g\, (\mu(Y_B^{gsT}(c + 1,d)) - \mu(Y_B^{gsT}(c,d + 1)))^2$

Boundary Values

$\sigma^2(Y_A^{gsT}(c,d)) = 0$, if $c = 6$ and $0 \le d \le 4$; $d = 6$ and $0 \le c \le 4$; (7,5); (5,7)

$\sigma^2(Y_A^{gsT}(6,6)) = 0$

$\sigma^2(Y_A^{gsT}(a,b : c,d))$

$= P_A^{pg}(a,b)\sigma^2(Y_B^{gsT}(c + 1,d)) + (1 - P_A^{pg}(a,b))\sigma^2(Y_B^{gsT}(c,d + 1)) + P_A^{pg}(a,b)\, (1-P_A^{pg}(a,b))\, (\mu(Y_B^{gsT}(c + 1,d)) - \mu(Y_B^{gsT}(c,d + 1)))^2$, if $(c,d) \neq (6,6)$

$\sigma^2(Y_A^{gsT}(a,b : c,d)) = 0$, if $(c,d) = (6,6)$

Table 15 represents the variance of the number of games remaining in a tiebreak set from various score lines with player A currently serving with $p_A = 0.62$ and $p_B = 0.60$. It indicates that the variance of the number of games remaining from the outset in such a set is 3.7.

					B score				
		0	1	2	3	4	5	6	7
	0	3.7	3.5	3.6	4.2	3.2	1.8	0	
	1	3.7	3.3	3.1	3.2	3.6	1.6	0	
	2	4.1	3.2	2.9	2.8	2.6	2.2	0	
A score	3	3.3	3.6	2.7	2.5	2.4	1.6	0	
	4	2.5	2.1	2.7	1.9	1.8	1.4	0	
	5	1.0	1.3	1.1	1.8	1.3	0.2	0.2	0
	6	0	0	0	0	0	0.2	0	
	7						0		

Table 15: The variance of the number of games remaining in a tiebreak set from various score lines with player A currently serving with $p_A = 0.62$ and $p_B = 0.60$

Recurrence Formula

$\mu(Y_A^{gs}(c,d)) = 1 + p_A^g\, \mu(Y_B^{gs}(c + 1,d)) + q_A^g\, \mu(Y_B^{gs}(c,d + 1))$

Boundary Values

$\mu(Y_A^{gs}(c,d)) = 0$, if $c = 6$ and $0 \le d \le 4$; $d = 6$ and $0 \le c \le 4$

$\mu(Y_A^{gs}(5,5)) = 2/(p_A^g\, q_B^g + q_A^g\, p_B^g)$

$\mu(Y_A^{gs}(a,b : c,d)) = 1 + P_A^{pg}(a,b)\, \mu(Y_B^{gs}(c + 1,d)) + (1 - P_A^{pg}(a,b))\, \mu(Y_B^{gs}(c,d + 1))$

Table 16 represents the mean number of games remaining in an advantage set from various score lines with player A currently serving with $p_A = 0.62$ and $p_B = 0.60$. It indicates that the mean number of games remaining from the outset in such a set is 10.8.

					B score			
		0	1	2	3	4	5	6
	0	10.8	10.0	8.6	6.9	4.5	2.4	0
	1	9.4	9.2	8.4	6.8	5.0	2.4	0
	2	7.8	7.7	7.7	6.9	4.9	2.8	0
A score	3	5.6	6.0	5.9	6.3	5.7	2.9	0
	4	3.7	3.7	4.2	4.3	5.4	5.2	0
	5	1.5	1.6	1.6	2.1	2.2	5.4	
	6	0	0	0	0	0		

Table 16: The mean number of games remaining in an advantage set from various score lines with player A currently serving with $p_A = 0.62$ and $p_B = 0.60$

Recurrence Formula

$\sigma^2(Y_A^{gs}(c,d)) = p_A^g\, \sigma^2(Y_B^{gs}(c+1,d)) + q_A^g\, \sigma^2(Y_B^{gs}(c,d+1)) + p_A^g\, q_A^g(\mu(Y_B^{gs}(c+1,d)) - \mu(Y_B^{gs}(c,d+ 1)))^2$

Boundary Values

$\sigma^2(Y_A^{gs}(c,d)) = 0$, if $c = 6$ and $0 \le d \le 4$; $d = 6$ and $0 \le c \le 4$

$\sigma^2(Y_A^{gs}(5,5)) = 4(p_A^g p_B^g + q_A^g q_B^g)/(p_A^g q_B^g + q_A^g p_B^g)^2$

$\sigma^2(Y_A^{gs}(a,b : c,d)) = P_A^{pg}(a,b)\sigma^2(Y_B^{gs}(c+1,d)) + (1-P_A^{pg}(a,b))\sigma^2(Y_B^{gs}(c,d+1)) + P_A^{pg}(a,b)(1-P_A^{pg}(a,b))(\mu(Y_B^{gs}(c+1,d)) - \mu(Y_B^{gs}(c,d+1)))^2$

Table 17 represents the variance of the number of games remaining in an advantage set from various score lines with player A currently serving with $p_A = 0.62$ and $p_B = 0.60$. It indicates that the variance of the number of games remaining from the outset in such a set is 14.8.

					B score			
		0	1	2	3	4	5	6
	0	14.8	15.1	13.7	13.7	7.1	3.6	0
	1	13.1	15.1	15.6	13.3	12.4	3.7	0
	2	13.1	12.9	15.8	16.5	12.3	9.0	0
A score	3	7.6	12.5	12.5	17.0	18.2	9.2	0
	4	5.2	5.4	11.2	11.5	18.4	19.4	0
	5	1.4	2.8	2.9	8.6	9.2	18.4	
	6	0	0	0	0	0		

Table 17: The variance of the number of games remaining in an advantage set from various score lines with player A currently serving with $p_A = 0.62$ and $p_B = 0.60$

Similar formulas can be obtained for $\mu(Y_B^{gsT}(c,d))$, $\mu(Y_B^{gs}(c,d))$, $\mu(Y_B^{gsT}(a,b : c,d))$, $\mu(Y_B^{gs}(a,b : c,d))$, $\sigma^2(Y_B^{gsT}(c,d))$, $\sigma^2(Y_B^{gs}(c,d))$, $\sigma^2(Y_B^{gsT}(a,b : c,d))$ and $\sigma^2(Y_B^{gs}(a,b : c,d))$

3.5 Notation for sets remaining in a match

Let $Y^{sm3}(e,f)$ represent a random for the number of sets remaining in a best-of-3 set match from set score (e,f).

Let $Y^{sm5}(e,f)$ represent a random for the number of sets remaining in a best-of-5 set match from set score (e,f).

Let $Y_A^{sm3}(a,b : c,d : e,f)$ and $Y_B^{sm3}(a,b : c,d : e,f)$ represent random variables for the number of sets remaining in a best-of-3 set match from point, game and set score (a,b : c,d : e,f) for player A and player B serving respectively.

Let $Y_A^{sm5}(a,b : c,d : e,f)$ and $Y_B^{sm5}(a,b : c,d : e,f)$ represent random variables for the number of sets remaining in a best-of-3 set match from point, game and set score (a,b : c,d : e,f) for player A and player B serving respectively.

Let $\mu(Y^{sm3}(e,f))$ represent the mean number of sets remaining in a best-of-3 set match from set score (e,f).

Let $\mu(Y^{sm5}(e,f))$ represent the mean number of sets remaining in a best-of-5 set match from set score (e,f).

Let $\mu(Y_A^{sm3}(a,b : c,d : e,f))$ and $\mu(Y_B^{sm3}(a,b : c,d : e,f))$ represent the mean number of sets remaining in a best-of-3 set match from point, game and set score (a,b : c,d : e,f) for player A and player B serving respectively.

Let $\mu(Y_A^{sm5}(a,b : c,d : e,f))$ and $\mu(Y_B^{sm5}(a,b : c,d : e,f))$ represent the mean number of sets remaining in a best-of-5 set match from point, game and set score (a,b : c,d : e,f) for player A and player B serving respectively.

Let $\sigma^2(Y^{sm3}(e,f))$ represent the variance of the number of sets remaining in a best-of-3 set match from set score (e,f).

Let $\sigma^2(Y^{sm5}(e,f))$ represent the variance of the number of sets remaining in a best-of-5 set match from set score (e,f).

Let $\sigma^2(Y_A^{sm3}(a,b : c,d : e,f))$ and $\sigma^2(Y_B^{sm3}(a,b : c,d : e,f))$ represent the variance of the number of sets remaining in a best-of-3 set match from point, game and set score (a,b : c,d : e,f) for player A and player B serving respectively.

Let $\sigma^2(Y_A^{sm5}(a,b : c,d : e,f))$ and $\sigma^2(Y_B^{sm5}(a,b : c,d : e,f))$ represent the variance of the number of sets remaining in a best-of-5 set match from point, game and set score (a,b : c,d : e,f) for player A and player B serving respectively.

3.6 Mean and variance of sets remaining in a match

Recurrence Formula
$\mu(Y^{sm3}(e,f)) = 1 + p^{sT}\mu(Y^{sm3}(e + 1,f)) + q^{sT}\mu(Y^{sm3}(e,f + 1))$

Boundary Values
$\mu(Y^{sm3}(e,f)) = 0$, if $e = 2$ and $f \leq 1$; $f = 2$ and $e \leq 1$
$\mu(Y^{sm3}(1,1)) = 1$

Recurrence Formula
$\mu(Y^{sm5}(e,f)) = 1 + p^{sT}\mu(Y^{sm5}(e + 1,f)) + q^{sT}\mu(Y^{sm5}(e,f + 1))$

Boundary Values
$\mu(Y^{sm5}(e,f)) = 0$, if $e = 3$ and $f \leq 1$; $f = 3$ and $e \leq 1$
$\mu(Y^{sm5}(2,2)) = 1$

$\mu(Y_A^{sm3}(a,b : c,d : e,f)) = 1 + P_A^{psT}(a,b : c,d)\mu(Y^{sm3}(e + 1,f)) + (1 - P_A^{psT}(a,b : c,d))\mu(Y^{sm3}(e,f + 1))$, if $(e,f) \neq (1,1)$
$\mu(Y_A^{sm3}(a,b : c,d : e,f)) = 1$, if $(e,f) = (1,1)$

$\mu(Y_A^{sm5}(a,b : c,d : e,f)) = 1 + P_A^{psT}(a,b : c,d)\mu(Y^{sm5}(e + 1,f)) + (1 - P_A^{psT}(a,b : c,d))\mu(Y^{sm5}(e,f + 1))$, if $(e,f) \neq (2,2)$
$\mu(Y_A^{sm5}(a,b : c,d : e,f)) = 1$, if $(e,f) = (2,2)$

Table 18 represents the mean number of sets remaining in a best-of-5 set match from various score lines with $p_A = 0.62$ and $p_B = 0.60$. It indicates that the mean number of sets remaining from the outset in such a match is 4.1.

			B score		
		0	1	2	3
	0	4.1	3.2	1.9	0
A score	1	3.0	2.5	1.6	0
	2	1.6	1.4	1	
	3	0	0		

Table 18: The mean number of sets remaining in a best-of-5 set match from various score lines with $p_A = 0.62$ and $p_B = 0.60$

Recurrence Formula
$\sigma^2(Y^{sm3}(e,f)) = p^{sT}\sigma^2(Y^{sm3}(e+1,f))+q^{sT}\sigma^2(Y^{sm3}(e,f+1))+p^{sT}q^{sT}(\mu(Y^{sm3}(e+1,f)) - \mu(Y^{sm3}(e,f + 1)))^2$

Boundary Values
$\sigma^2(Y^{sm3}(e,f)) = 0$, if $e = 2$ and $f \leq 1$; $f = 2$ and $e \leq 1$
$\sigma^2(Y^{sm5}(1,1)) = 0$

Recurrence Formula
$\sigma^2(Y^{sm5}(e,f)) = p^{sT}\sigma^2(Y^{sm5}(e+1,f))+q^{sT}\sigma^2(Y^{sm5}(e,f+1))+p^{sT}q^{sT}(\mu(Y^{sm5}(e+1,f)) - \mu(Y^{sm5}(e,f + 1)))^2$

Boundary Values
$\sigma^2(Y^{sm5}(e,f)) = 0$, if $e = 3$ and $f \leq 1$; $f = 3$ and $e \leq 1$
$\sigma^2(Y^{sm5}(2,2)) = 0$

$\sigma^2(Y_A^{sm3}(a,b : c,d : e,f)) = P_A^{psT}(a,b : c,d)\sigma^2(Y^{sm3}(e + 1,f)) + (1-P_A^{psT}(a,b : c,d))\, \sigma^2(Y^{sm3}(e,f + 1)) + P_A^{psT}(a,b : c,d)(1-P_A^{psT}(a,b : c,d))(\mu(Y^{sm3}(e + 1,f)) - \mu(Y^{sm3}(e,f + 1)))^2$, if $(e,f) \neq (1,1)$
$\sigma^2(Y_A^{sm3}(a,b : c,d : e,f)) = 0$, if $(e,f) = (1,1)$

$\sigma^2(Y_A^{sm5}(a,b : c,d : e,f)) = P_A^{psT}(a,b : c,d)\sigma^2(Y^{sm5}(e + 1,f)) + (1-P_A^{psT}(a,b : c,d))\, \sigma^2(Y^{sm5}(e,f + 1)) + P_A^{psT}(a,b : c,d)(1-P_A^{psT}(a,b : c,d))(\mu(Y^{sm5}(e + 1,f)) - \mu(Y^{sm5}(e,f + 1)))^2$, if $(e,f) \neq (2,2)$
$\sigma^2(Y_A^{sm5}(a,b : c,d : e,f)) = 0$, if $(e,f) = (2,2)$

Table 19 represents the variance of the number of sets remaining in a best-of-5 set match from various score lines with $p_A = 0.62$ and $p_B = 0.60$. It indicates that the variance of the number of sets remaining from the outset in such a match is 0.62.

			B score		
		0	1	2	3
	0	0.62	0.55	0.74	0
A score	1	0.64	0.25	0.25	0
	2	0.61	0.25	0	
	3	0	0		

Table 19: The variance of the number of sets remaining in a best-of-5 set match from various score lines with $p_A = 0.62$ and $p_B = 0.60$

Similar formulas can be obtained for $\mu(Y_B^{sm3}(a,b : c,d : e,f))$, $\mu(Y_B^{sm3T}(a,b : c,d : e,f))$, $\sigma^2(Y_B^{sm3}(a,b : c,d : e,f))$ and $\sigma^2(Y_B^{sm3T}(a,b : c,d : e,f))$.

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Chapter 2: Badminton

1. Scoring Systems

To win a game requires winning 21 points. However, if the scores are level at 20-all then play continues until a player has established a two-point lead and wins the game. Further, if the score reaches 29-all then the player to win the next point wins the game. The winner of each point is the server for the next point in a game.

To win a match requires winning 2 games. The winner of a game serves the first point of the next game. Usually the toss of a coin decides the first server in the match.

2. Chances of Winning

2.1 Notation for winning a game

Let p_A represent the probability of player A winning a point on serve
Let p_B represent the probability of player B winning a point on serve

Let q_A=1-p_A and q_B=1-p_B

Let $P_A^{pg}(a,b)$ and $P_B^{pg}(a,b)$ represent the probabilities of player A winning a game at point score (a,b) for player A and player B serving respectively

Let $P_A^{pg}(0,0) = p_A^g$
Let $P_B^{pg}(0,0)=q_B^g$

Let q_A^g=1- p_A^g and p_B^g=1-q_B^g

2.2 Winning a game

Recurrence Formulas

$P_A^{pg}(a,b) = p_A P_A^{pg}(a+1,b)+q_A P_B^{pg}(a,b+1)$

$P_B^{pg}(a,b) = p_B P_B^{pg}(a,b+1) +q_B P_A^{pg}(a+1,b)$

Boundary Values

$P_A^{pg}(a,b) = P_B^{pg}(a,b) = 1$ if a=21 and b ≤19 or (a,b) = (22,20), (23,21), (24,22), (25,23), (26,24), (27,25), (28,26), (29,27), (30,28), (30,29)

$P_A^{pg}(a,b) = P_B^{pg}(a,b) = 0$, if b=21 and a ≤19 or (a,b) = (20,22), (21,23), (22,24), (23,25), (24,26), (25,27), (26,28), (27,29), (28,30), (29,30)

$P_A^{pg}(29,29) = p_A$

$P_B^{pg}(29,29) = q_B$

2.3 Notation for winning a match

Let $P_A^{gm}(c,d)$ and $P_B^{gm}(c,d)$ represent the probabilities of player A winning a match at game score (c,d) for player A and player B serving respectively

Let Let $P_A^{pm}(a,b:c,d)$ and $P_B^{pm}(a,b:c,d)$ represent the probabilities of player A winning a match at point and game score (a,b:c,d) for player A and player B serving respectively

2.4 Winning a match

Recurrence Formulas

$P_A^{gm}(c,d)= p_A^g P_A^{gm}(c+1,d)+q_A^g P_B^{gm}(c,d+1)$

$P_B^{gm}(c,d)=q_B^g P_A^{gm}(c+1,d)+p_B^g P_B^{gm}(c,d+1)$

Boundary Values

$P_A^{gm}(2,0)=P_B^{gm}(2,0)=1$

$P_A^{gm}(0,2)=P_B^{gm}(0,2)=0$

$P_A^{gm}(1,1)= p_A^g$

$P_B^{gm}(1,1)= q_B^g$

$P_A^{pm}(a,b:c,d)= P_A^{pg}(a,b)P_A^{gm}(c+1,d)+(1- P_A^{pg}(a,b))P_B^{gm}(c,d+1)$

$P_B^{pm}(a,b:c,d)= P_B^{pg}(a,b) P_A^{gm}(c+1,d)+(1-P_B^{pg}(a,b)) P_B^{gm}(c,d+1)$

Table 1 exhibits the probabilities of player A winning a game and match for different values of p_A and p_B, and for both player A and player B serving first. It can be observed from the table that when p_A and p_B are greater than 0.5, it is an advantage to serve first in the match. Similarly, when p_A and p_B are less than 0.5, it is an advantage to receive first in the match.

p_A	p_B	$P_A^{pg}(0,0)$	$P_B^{pg}(0,0)$	$P_A^{gm}(0,0)$	$P_B^{gm}(0,0)$
0.46	0.46	0.495	0.505	0.497	0.503
0.48	0.46	0.549	0.556	0.577	0.581
0.50	0.46	0.602	0.607	0.653	0.656
0.52	0.46	0.653	0.655	0.723	0.725
0.48	0.48	0.497	0.503	0.499	0.501
0.50	0.48	0.551	0.554	0.578	0.579
0.52	0.48	0.604	0.604	0.654	0.654
0.50	0.50	0.500	0.500	0.500	0.500
0.52	0.50	0.554	0.551	0.579	0.578
0.52	0.52	0.503	0.497	0.501	0.499

Table 1: Probabilities of player A winning a game and match for different values of p_A and p_B, and for both player A and player B serving first

3. Match Duration

3.1 Notation for points remaining in a game

Let $Y_A^{pg}(a,b)$ and $Y_B^{pg}(a,b)$ represent random variables of the number of points remaining in a game at point score (a,b) for player A and player B serving respectively.

Let $\mu(Y_A^{pg}(a,b))$ and $\mu(Y_B^{pg}(a,b))$ represent the mean number of points remaining in a game at point score (a,b) for player A and player B serving respectively

Let $\sigma^2(Y_A^{pg}(a,b))$ and $\sigma^2(Y_B^{pg}(a,b))$ represent the variance of the number of points remaining in a game at point score (a,b) for player A and B serving respectively

3.2 Mean and variance of points remaining in a game

Recurrence Formulas

$\mu(Y_A^{pg}(a,b))=1+p_A\,\mu(Y_A^{pg}(a+1,b))+q_A\,\mu(Y_B^{pg}(a,b+1))$

$\mu(Y_B^{pg}(a,b))=1+p_B\,\mu(Y_B^{pg}(a,b+1))+q_B\,\mu(Y_A^{pg}(a+1,b))$

Boundary Values

$\mu(Y_A^{pg}(a,b))=\mu(Y_B^{pg}(a,b))=0$, if a=21 and b ≤19 or b=21 and a ≤19 or (a,b) = (22,20), (23,21), (24,22), (25,23), (26,24), (27,25), (28,26), (29,27), (30,28), (20,22), (21,23), (22,24), (23,25), (24,26), (25,27), (26,28), (27,29), (28,30)

$\mu(Y_A^{pg}(29,29))=\mu(Y_B^{pg}(29,29))=1$

Recurrence Formulas

$\sigma^2(Y_A^{pg}(a,b))= p_A\,\sigma^2(Y_A^{pg}(a+1,b))+q_A\,\sigma^2(Y_B^{pg}(a,b+1))+p_Aq_A(\mu(Y_A^{pg}(a+1,b))-\mu(Y_B^{pg}(a,b+1)))^2$

$\sigma^2(Y_B^{pg}(a,b))= p_B\,\sigma^2(Y_B^{pg}(a,b+1))+q_B\,\sigma^2(Y_A^{pg}(a+1,b))+p_Bq_B(\mu(Y_B^{pg}(a,b+1))-\mu(Y_A^{pg}(a+1,b)))^2$

Boundary Values
$\sigma^2(Y_A^{pg}(a,b))=\sigma^2(Y_B^{pg}(a,b))=0$, if a=21 and b ≤19 or b=21 and a ≤19 or (a,b) = (22,20), (23,21), (24,22), (25,23), (26,24), (27,25), (28,26), (29,27), (30,28), (20,22), (21,23), (22,24), (23,25), (24,26), (25,27), (26,28), (27,29), (28,30)
$\sigma^2(Y_A^{pg}(29,29))=\sigma^2(Y_B^{pg}(29,29))=0$
$\sigma^2(Y_A^{pg}(a,b))=\sigma^2(Y_B^{pg}(a,b))=0$, if a=21 and b ≤19 or b=21 and a ≤19 or (a,b) = (22,20), (23,21), (24,22), (25,23), (26,24), (27,25), (28,26), (29,27), (30,28), (20,22), (21,23), (22,24), (23,25), (24,26), (25,27), (26,28), (27,29), (28,30)
$\sigma^2(Y_A^{pg}(29,29))=\sigma^2(Y_B^{pg}(29,29))=0$

Table 2 represents the mean and standard deviation of the number of points remaining in a game for different values of p_A and p_B, for both player A and player B serving first.

p_A	p_B	$\mu(Y_A^{pg}(0,0))$	$\sigma(Y_A^{pg}(0,0))$	$\mu(Y_B^{pg}(0,0))$	$\sigma(Y_B^{pg}(0,0))$
0.46	0.46	37.63	3.91	37.63	3.91
0.48	0.46	37.51	3.92	37.50	3.92
0.50	0.46	37.32	3.95	37.30	3.95
0.52	0.46	37.06	4.00	37.05	4.00
0.48	0.48	37.43	3.92	37.43	3.92
0.50	0.48	37.31	3.93	37.30	3.94
0.52	0.48	37.11	3.98	37.11	3.98
0.50	0.50	37.24	3.94	37.24	3.94
0.52	0.50	37.10	3.96	37.10	3.96
0.52	0.52	37.03	3.97	37.03	3.97

Table 2: Mean and standard deviation of the number of points remaining in a game for different values of p_A and p_B, for both player A and player B serving first

3.3 Notation for games remaining in a match

Let $Y_A^{gm}(c,d)$ and $Y_B^{gm}(c,d)$ represent random variables for the number of games remaining in a match from game score (c,d) for player A and player B serving respectively.

Let $Y_A^{gm}(a,b:c,d)$ and $Y_B^{gm}(a,b:c,d)$ represent random variables for the number of games remaining in a match from point and game score (a,b:c,d) for player A and player B serving respectively.

Let $\mu(Y_A^{gm}(c,d))$ and $\mu(Y_B^{gm}(c,d))$ represent the mean number of games remaining in a match at game score (c,d) for player A and player B serving respectively.

Let $\mu(Y_A^{gm}(a,b:c,d))$ and $\mu(Y_B^{gm}(a,b:c,d))$ represent the mean number of games remaining in a match at point and game score (a,b:c,d) for player A and player B serving respectively.

Let $\sigma^2(Y_A^{gm}(c,d))$ and $\sigma^2(Y_B^{gm}(c,d))$ represent the variance of the number of games remaining in a match at game score (c,d) for player A and player B serving respectively.

Let $\sigma^2(Y_A^{gm}(a,b:c,d))$ and $\sigma^2(Y_B^{gm}(a,b:c,d))$ represent the variance of the number of games remaining in a match at point and game score (a,b:c,d) for player A and player B serving respectively.

3.4 Mean and variance of games remaining in a match

Recurrence Formulas

$\mu(Y_A^{gm}(c,d))= 1+p_A^g\ \mu(Y_A^{gm}(c+1,d))+q_A^g\ \mu(Y_B^{gm}(c,d+1))$

$\mu(Y_B^{gm}(c,d))=1+q_B^g\ \mu(Y_A^{gm}(c+1,d))+p_B^g\ \mu(Y_B^{gm}(c,d+1))$

Boundary Values

$\mu(Y_A^{gm}(2,0))= \mu(Y_A^{gm}(0,2))= \mu(Y_B^{gm}(2,0))= \mu(Y_B^{gm}(0,2))=0$

$\mu(Y_A^{gm}(1,1))= \mu(Y_B^{gm}(1,1))=1$

$\mu(Y_A^{gm}(a,b:c,d))=1+P_A^{pg}(a,b)\mu(Y_A^{gm}(c+1,d))+(1-P_A^{pg}(a,b))\mu(Y_B^{gm}(c,d+1))$

$\mu(Y_B^{gm}(a,b:c,d))=1+P_B^{pg}(a,b)\ \mu(Y_A^{gm}(c+1,d))+(1-P_B^{pg}(a,b))\mu(Y_B^{gm}(c,d+1))$

Recurrence Formulas:

$\sigma^2(Y_A^{gm}(c,d))=p_A^g\ \sigma^2(Y_A^{gm}(c+1,d))+q_A^g\ \sigma^2(Y_B^{gm}(c,d+1))+p_A^g\ q_A^g(\mu(Y_A^{gm}(c+1,d))- \mu(Y_B^{gm}(c,d+1)))^2$

$\sigma^2(Y_B^{gm}(c,d))= q_B^g\ \sigma^2(Y_A^{gm}(c+1,d))+p_B^g\ \sigma^2(Y_B^{gm}(c,d+1))+ q_B^g p_B^g(\mu(Y_A^{gm}(c+1,d))- \mu(Y_B^{gm}(c,d+1)))^2$

Boundary Values

$\sigma^2(Y_A^{gm}(2,0))= \sigma^2(Y_A^{gm}(0,2))= \sigma^2(Y_B^{gm}(2,0))= \sigma^2(Y_B^{gm}(0,2))=0$

$\sigma^2(Y_A^{gm}(1,1))= \sigma^2(Y_B^{gm}(1,1))=0$

$\sigma^2(Y_A^{gm}(a,b:c,d))=P_A^{pg}(a,b)\ \sigma^2(Y_A^{gm}(c+1,d))+(1-P_A^{pg}(a,b))\ \sigma^2(Y_B^{gm}(c,d+1))+P_A^{pg}(a,b)\ (1-P_A^{pg}(a,b))$
$(\mu(Y_A^{gm}(c+1,d))-\mu(Y_B^{gm}(c,d+1)))^2$

$\sigma^2(Y_B^{gm}(a,b:c,d))= P_B^{pg}(a,b)\ \sigma^2(Y_A^{gm}(c+1,d))+(1-P_B^{pg}(a,b))\ \sigma^2(Y_B^{gm}(c,d+1))+P_B^{pg}(a,b)(1-P_B^{pg}(a,b))$
$(\mu(Y_A^{gm}(c+1,d))-\mu(Y_B^{gm}(c,d+1)))^2$

Table 3 represents the mean and standard deviation of the number of games remaining in a match for different values of p_A and p_B, for both player A and player B serving first.

p_A	p_B	$\mu(Y_A^{gm}(0,0))$	$\sigma(Y_A^{gm}(0,0))$	$\mu(Y_B^{gm}(0,0))$	$\sigma(Y_B^{gm}(0,0))$
0.46	0.46	2.51	0.50	2.51	0.50
0.48	0.46	2.50	0.50	2.50	0.50
0.50	0.46	2.48	0.50	2.48	0.50
0.52	0.46	2.45	0.50	2.45	0.50
0.48	0.48	2.50	0.50	2.50	0.50
0.50	0.48	2.50	0.50	2.50	0.50
0.52	0.48	2.48	0.50	2.48	0.50
0.50	0.50	2.50	0.50	2.50	0.50
0.52	0.50	2.49	0.50	2.49	0.50
0.52	0.52	2.50	0.50	2.50	0.50

Table 3: Mean and standard deviation of the number of games remaining in a match for different values of p_A and p_B, for both player A and player B serving first

4. References

Barnett T (2014). Modelling outcomes in badminton. Strategic Games.

Chapter 3: Volleyball

1. Scoring Systems

There are two types of volleyball. Standard volleyball known as just volleyball and beach volleyball. The scoring systems for each type are very similar and only the formulas for volleyball are given.

To win an early set in volleyball requires winning 25 points. However, if the scores are level at 24 points-all then play continues indefinitely until a player has established a two-point lead and wins the set. If the match reaches a final deciding fifth set, then to win the match requires winning 15 points. However, if the scores are level at 14 points-all then play continues indefinitely until a player has established a two-point lead and wins the match. The server in the set is decided by the winner of the previous point.

To win an early set in beach volleyball requires winning 21 points. However, if the scores are level at 19 points-all then play continues indefinitely until a player has established a two-point lead and wins the set. If the match reaches a final deciding fifth set, then to win the match requires winning 15 points. However, if the scores are level at 14 points-all then play continues indefinitely until a player has established a two-point lead and wins the match. The server in the set is decided by the winner of the previous point.

To win a match in volleyball requires winning 3 sets. Usually the toss of a coin is used to decide the first server of the match. The first server in subsequent sets is rotated, unless the set score reaches 2 sets-all, where the toss of a coin is used to decide the first server in the final set.

To win a match in beach volleyball requires winning 2 sets. Usually the toss of a coin is used to decide the first server of the match. The first server is subsequent sets is rotated, unless the set score reaches 1 set-all, where the toss of a coin is used to decide the first server in the final set.

Volleyball is more complex to analyze than tennis and badminton due to the rotation on serve which is why the recurrence formulas for volleyball are given in this chapter. The formulas for table tennis and squash can readily be derived from the formulas given in tennis and badminton. A calculator featuring badminton, volleyball, beach volleyball, table tennis and squash can be freely downloaded from (xlsx). A summary of the different scoring systems used in these sports in given below.

Sport	**Match**	**Game/Set**	**Final Set**	**Server in game/set**	**Serving first each game/set**
Badminton	Best-of-3 games	First to 21 points At 20-all, 2 point lead At 29-all, next point		Winner of point	Winner of game
Table Tennis	Best-of 5 games	First to 11 points At 10-all, 2 point lead		Rotates every two points Rotates every point at 10-all	Rotate server
Volleyball	Best-of 5 sets	First to 25 points At 24-all, 2 point lead	First to 15 points At 14-all, 2 point lead	Winner of point	Rotate server Coin toss at 2 sets-all
Beach Volleyball	Best-of 3 sets	First to 21 points At 20-all, 2 point lead	First to 15 points At 14-all, 2 point lead	Winner of point	Rotate server Coin toss at 1 sets-all
Squash	Best-of-5 games	First to 11 points At 10-all, 2 point lead		Winner of point	Winner of game

2. Chances of Winning

2.2 Notation for a winning a set

Let p_A represent the probability of player A winning a point on serve
Let p_B represent the probability of player B winning a point on serve

Let $q_A=1-p_A$ and $q_B=1-p_B$

Let $P_A^{25ps}(a,b)$ and $P_B^{25ps}(a,b)$ represent the probabilities of player A winning a first to 25 point set at point score (a,b) for player A and player B serving respectively.

Let $P_A^{15ps}(a,b)$ and $P_B^{15ps}(a,b)$ represent the probabilities of player A winning a first to 15 point set at point score (a,b) for player A and player B serving respectively

Let $P_A^{25ps}(0,0) = p_A^{25s}$
Let $P_B^{25ps}(0,0)=q_B^{25s}$
Let $P_A^{15ps}(0,0) = p_A^{15s}$
Let $P_B^{15ps}(0,0)=q_B^{15s}$

Let $q_A^{25s}=1- p_A^{25s}$ and $p_B^{25s}=1-q_B^{25s}$
Let $q_A^{15s}=1- p_A^{15s}$ and $p_B^{15s}=1-q_B^{15s}$

2.3 Winning a set

Recurrence Formulas
$P_A^{25ps}(a,b)=p_AP_A^{25ps}(a+1,b)+q_AP_B^{25ps}(a,b+1)$
$P_B^{25ps}(a,b)=p_BP_B^{25ps}(a,b+1)+q_BP_A^{25ps}(a+1,b)$

Boundary Values
$P_A^{25ps}(a,b)= P_B^{25ps}(a,b)=1$, if $a=25$, $0\leq b\leq 23$
$P_A^{25ps}(a,b)= P_B^{25ps}(a,b)=0$, if $b=25$, $0\leq a\leq 23$
$P_A^{25ps}(24,24) = p_A^2/((1-q_Aq_B)^2-p_Aq_Ap_Bq_B)$
$P_B^{25ps}(24,24) = p_Aq_B(1+p_Ap_B-q_Aq_B)/((1-q_Aq_B)^2-p_Aq_Ap_Bq_B)$

Recurrence Formulas
$P_A^{15ps}(a,b)=p_AP_A^{15ps}(a+1,b)+q_AP_B^{15ps}(a,b+1)$
$P_B^{15ps}(a,b)=p_BP_B^{15ps}(a,b+1)+q_BP_A^{15ps}(a+1,b)$

Boundary Values
$P_A^{15ps}(a,b) = P_A^{15ps}(a,b)=1$, if $a=15$, $0\leq b\leq 13$
$P_B^{15ps}(a,b) = P_B^{15ps}(a,b)=0$, if $b=15$, $0\leq a\leq 13$
$P_A^{15ps}(14,14) = p_A^2/((1-q_Aq_B)^2-p_Aq_Ap_Bq_B)$
$P_B^{15ps}(14,14) = p_Aq_B(1+p_Ap_B-q_Aq_B)/((1-q_Aq_B)^2-p_Aq_Ap_Bq_B)$

Table 1 represents the probability of team A winning a 25-point and 15-point set for different values of p_A and p_B from the start of the set. The average probability of winning points on serve in men's volleyball is about 0.25. Therefore, the values of p_A and p_B were chosen to reflect this value. The results indicate that the team receiving first has an advantage in winning the set. This is not an unexpected result, since the receiving team has the first opportunity at an attack.

	25-point set		15-point set	
p_A, p_B	A serving	B serving	A serving	B serving
0.30, 0.30	0.48	0.52	0.47	0.53
0.30, 0.29	0.51	0.56	0.49	0.56
0.30, 0.25	0.63	0.69	0.59	0.66
0.25, 0.25	0.47	0.53	0.46	0.54
0.25, 0.24	0.50	0.57	0.48	0.57
0.20, 0.20	0.46	0.54	0.45	0.55

Table 1: The probability of team A winning a first to 25-points and first to 15-points set for different values of p_A and p_B from the start of the set

2.4 Notation for winning a match

Let $P_A^{sm}(c,d)$ and $P_B^{sm}(c,d)$ represent the probabilities of player A winning a match at set score (c,d) for player A and player B serving respectively

Let $P_A^{pm}(a,b:c,d)$ and $P_B^{pm}(a,b:c,d)$ represent the probabilities of player A winning a match at point and set score (a,b:c,d) for player A and player B serving respectively

2.2 Winning a match

Recurrence Formulas

$P_A^{sm}(c,d)= p_A^{25s}P_B^{sm}(c+1,d) +q_A^{25s}P_B^{sm}(c,d+1)$, if (c,d)=(0,0),(1,0),(0,1),(2,0),(0,2),(1,1)

$P_A^{sm}(c,d)= p_A^{25s}P_B^{sm}(c+1,d) +q_A^{25s}0.5(P_B^{sm}(c,d+1)+ P_A^{sm}(c,d+1))$, if (c,d)=(2,1)

$P_A^{sm}(c,d)= p_A^{25s}0.5(P_B^{sm}(c+1,d)+P_A^{sm}(c+1,d)) +q_A^{25s}P_B^{sm}(c,d+1)$, if (c,d)=(1,2)

$P_B^{sm}(c,d)= q_B^{25s} P_A^{sm}(c+1,d)+p_B^{25s} P_A^{sm}(c,d+1)$, if (c,d)=(0,0),(1,0),(0,1),(2,0),(0,2),(1,1)

$P_B^{sm}(c,d)= q_B^{25s} P_A^{sm}(c+1,d)+p_B^{25s} 0.5(P_A^{sm}(c,d+1)+ P_B^{sm}(c,d+1))$, if (c,d)=(2,1)

$P_B^{sm}(c,d)= q_B^{25s} 0.5(P_A^{sm}(c+1,d)+ P_B^{sm}(c+1,d))+p_B^{25s}P_A^{sm}(c,d+1)$, if (c,d)=(1,2)

Boundary Values

$P_A^{sm}(c,d)= P_B^{sm}(c,d)=1$, if (c,d)=(3,0),(3,1)

$P_A^{sm}(c,d)= P_B^{sm}(c,d)=0$, if (c,d)=(0,3),(1,3)

$P_A^{sm}(2,2)= p_A^{15s}$

$P_B^{sm}(2,2)= q_B^{15s}$

$P_A^{pm}(a,b:c,d)=P_A^{25ps}(a,b)P_A^{sm}(c+1,d) +(1-P_A^{25ps}(a,b))P_B^{sm}(c,d+1)$, if $(c,d)\neq(2,2)$

$P_A^{pm}(a,b:c,d)=P_A^{15ps}(a,b)$, if (c,d)=(2,2)

$P_B^{pm}(a,b:c,d)= P_B^{25ps}(a,b)P_A^{sm}(c+1,d) +(1-P_B^{25ps}(a,b))P_B^{sm}(c,d+1)$, if $(c,d)\neq(2,2)$

$P_B^{pm}(a,b:c,d)= P_B^{15ps}(a,b)$, if (c,d)=(2,2)

Table 2 represents the probability of team A winning a match for different values of p_A and p_B from the start of the match. The results indicate that there is no advantage in serving or receiving at the start of the match. At the start of the fifth set, another coin toss is used to determine the serving team, and as given in Table 1, it is an advantage for the remainder of the match to be receiving first in this final set.

	match	
p_A, p_B	A serving	B serving
0.30, 0.30	0.50	0.50
0.30, 0.29	0.56	0.56
0.30, 0.25	0.77	0.77
0.25, 0.25	0.50	0.50
0.25, 0.24	0.56	0.56
0.20, 0.20	0.50	0.50

Table 2: The probability of team A winning a match for different values of p_A and p_B from the start of the match

3. Match Duration

3.1 Notation for points remaining in a set

Let $Y_A^{25ps}(a,b)$ and $Y_B^{25ps}(a,b)$ represent random variables of the number of points remaining in a first-to-25 point set at point score (a,b) for player A and player B serving respectively.

Let $Y_A^{15ps}(a,b)$ and $Y_B^{15ps}(a,b)$ represent random variables of the number of points remaining in a first-to-15 point set at point score (a,b) for player A and player B serving respectively.

Let $\mu(Y_A^{25ps}(a,b))$ and $\mu(Y_B^{25ps}(a,b))$ represent the mean number of points remaining in a first to 25 point set at point score (a,b) for player A and player B serving respectively.

Let $\mu(Y_A^{15ps}(a,b))$ and $\mu(Y_B^{15ps}(a,b))$ represent the mean number of points remaining in a first to 15 point set at point score (a,b) for player A and player B serving respectively.

Let $\sigma^2(Y_A^{25ps}(a,b))$ and $\sigma^2(Y_B^{25ps}(a,b))$ represent the variance of the number of points remaining in a first to 25 point set at point score (a,b) for player A and player B serving respectively.

Let $\sigma^2(Y_A^{15ps}(a,b))$ and $\sigma^2(Y_B^{15ps}(a,b))$ represent the variance of the number of points remaining in a first to 15 point set at point score (a,b) for player A and player B serving respectively

3.2 Mean and variance of points remaining in a set

Recurrence Formulas

$\mu(Y_A^{25ps}(a,b)) =1+p_A\, \mu(Y_A^{25ps}(a+1,b)) +q_A\, \mu(Y_B^{25ps}(a,b+1))$

$\mu(Y_B^{25ps}(a,b))=1+p_B\, \mu(Y_B^{25ps}(a,b+1))+q_B\, \mu(Y_A^{25ps}(a+1,b))$

Boundary Values
$\mu(Y_A^{25ps}(a,b)) = \mu(Y_B^{25ps}(a,b))=0$, if $a=25$, $0\leq b\leq 23$ or $b=25$, $0\leq a\leq 23$
$\mu(Y_A^{25ps}(24,24))= 2(1+p_Aq_A-q_Aq_B)/((1-q_Aq_B)^2-p_Aq_Ap_Bq_B)$
$\mu(Y_B^{25ps}(24,24))=2(1+p_Bq_B-q_Aq_B)/((1-q_Aq_B)^2-p_Aq_Ap_Bq_B)$

Recurrence Formulas
$\mu(Y_A^{15ps}(a,b)) =1+p_A\,\mu(Y_A^{15ps}(a+1,b)) +q_A\,\mu(Y_B^{15ps}(a,b+1))$
$\mu(Y_B^{15ps}(a,b))=1+p_B\,\mu(Y_B^{15ps}(a,b+1))+q_B\,\mu(Y_A^{15ps}(a+1,b))$

Boundary Values
$\mu(Y_A^{15ps}(a,b)) = \mu(Y_B^{15ps}(a,b))=0$, if $a=15$, $0\leq b\leq 13$ or $b=15$, $0\leq a\leq 13$
$\mu(Y_A^{15ps}(14,14))= 2(1+p_Aq_A-q_Aq_B)/((1-q_Aq_B)^2-p_Aq_Ap_Bq_B)$
$\mu(Y_B^{15ps}(14,14))=2(1+p_Bq_B-q_Aq_B)/((1-q_Aq_B)^2-p_Aq_Ap_Bq_B)$

Recurrence Formulas
$\sigma^2(Y_A^{25ps}(a,b))=p_A\sigma^2(Y_A^{25ps}(a+1,b))+q_A\sigma^2(Y_B^{25ps}(a,b+1))+p_Aq_A(\mu(Y_A^{25ps}(a+1,b))-\mu(Y_B^{25ps}(a,b+1)))^2$
$\sigma^2(Y_B^{25ps}(a,b))=p_B\sigma^2(Y_B^{25ps}(a,b+1))+q_B\sigma^2(Y_A^{25ps}(a+1,b))+p_Bq_B(\mu(Y_B^{25ps}(a,b+1))-\mu(Y_A^{25ps}(a+1,b)))^2$

Boundary Values
$\sigma^2(Y_A^{25ps}(a,b)) = \mu(Y_B^{25ps}(a,b))=0$, if $a=25$, $0\leq b\leq 23$ or $b=25$, $0\leq a\leq 23$
$\sigma^2(Y_A^{25ps}(24,24))=4q_A(p_A+q_B+3p_Ap_Bq_B-2q_Aq_B^2-p_A^2q_A-p_Ap_Bq_Aq_B^2-p_Aq_A^2q_B^2+p_A^2p_Bq_Aq_B+q_A^2q_B^3)/D^2$
$\sigma^2(Y_B^{25ps}(24,24))=4q_B(p_B+q_A+3p_Ap_Bq_A-2q_A^2q_B-p_B^2q_B-p_Ap_Bq_A^2q_B-p_Bq_A^2q_B^2+p_Ap_B^2q_Aq_B+q_A^3q_B^2)/D^2$

Recurrence Formulas
$\sigma^2(Y_A^{15ps}(a,b))=p_A\sigma^2(Y_A^{15ps}(a+1,b))+q_A\sigma^2(Y_B^{15ps}(a,b+1))+p_Aq_A(\mu(Y_A^{15ps}(a+1,b))-\mu(Y_B^{15ps}(a,b+1)))^2$
$\sigma^2(Y_B^{15ps}(a,b))=p_B\sigma^2(Y_B^{15ps}(a,b+1))+q_B\sigma^2(Y_A^{15ps}(a+1,b))+p_Bq_B(\mu(Y_B^{15ps}(a,b+1))-\mu(Y_A^{15ps}(a+1,b)))^2$

Boundary values:
$\sigma^2(Y_A^{15ps}(a,b)) = \mu(Y_B^{15ps}(a,b))=0$, if $a=15$, $0\leq b\leq 13$ or $b=15$, $0\leq a\leq 13$
$\sigma^2(Y_A^{25ps}(14,14))=4q_A(p_A+q_B+3p_Ap_Bq_B-2q_Aq_B^2-p_A^2q_A-p_Ap_Bq_Aq_B^2-p_Aq_A^2q_B^2+p_A^2p_Bq_Aq_B+q_A^2q_B^3)/D^2$
$\sigma^2(Y_B^{25ps}(14,14))=4q_B(p_B+q_A+3p_Ap_Bq_A-2q_A^2q_B-p_B^2q_B-p_Ap_Bq_A^2q_B-p_Bq_A^2q_B^2+p_Ap_B^2q_Aq_B+q_A^3q_B^2)/D^2$

where $D=(1-q_Aq_B)^2-p_Aq_Ap_Bq_B$

Table 3 gives numerical results of the number of points in a 25-point set for different values of p_A and p_B.

	Mean		Standard deviation	
p_A, p_B	A serving	B serving	A serving	B serving
0.30, 0.30	47.0	47.0	4.7	4.7
0.30, 0.29	47.1	47.1	4.8	4.8
0.30, 0.25	47.2	47.0	5.0	5.0
0.25, 0.25	47.8	47.8	5.4	5.4
0.25, 0.24	47.9	47.8	5.5	5.5
0.20, 0.20	48.9	48.9	6.7	6.7

Table 3: The mean and standard deviation of the number of points in a first to 25 points set for different values of p_A and p_B

3.3 Notation for sets remaining in a match

Let $\mu(Y_A^{sm}(a,b))$ and $\mu(Y_B^{sm}(a,b))$ and represent the mean number of sets remaining in a match at set score (c,d) for player A and player B serving respectively

Let $\mu(Y_A^{sm}(a,b:c,d))$ and $\mu(Y_B^{sm}(a,b:c,d))$ represent the mean number of sets remaining in a match at point and set score (a,b:c,d) for player A and player B serving respectively

Let $\sigma^2(Y_A^{sm}(a,b))$ and $\sigma^2(Y_B^{sm}(a,b))$ represent the variance of the number of sets remaining in a match at set score (c,d) for player A and player B serving respectively

Let $\sigma^2(Y_A^{sm}(a,b:c,d))$ and $\sigma^2(Y_B^{sm}(a,b:c,d))$ represent the variance of the number of sets remaining in a match at point and set score (a,b:c,d) for player A and player B serving respectively

3.4 Mean and variance of sets remaining in a match

Recurrence Formulas

$\mu(Y_A^{sm}(a,b))=1+p_A^{25s}\mu(Y_B^{sm}(c+1,d))+q_A^{25s}\mu(Y_B^{sm}(c,d+1))$

$\mu(Y_B^{sm}(a,b))=1+q_B^{25s}\mu(Y_A^{sm}(c+1,d))+p_B^{25s}\mu(Y_A^{sm}(c,d+1))$

Boundary Values

$\mu(Y_A^{sm}(a,b))=\mu(Y_B^{sm}(a,b))=0$, if (c,d)=(3,0),(3,1),(0,3),(1,3)

$\mu(Y_A^{sm}(2,2))=\mu(Y_B^{sm}(2,2))=1$

$\mu(Y_A^{sm}(a,b:c,d))=1+P_A^{25ps}(a,b)\mu(Y_B^{sm}(c+1,d))+(1-P_A^{25ps}(a,b))\mu(Y_B^{sm}(c,d+1))$

$\mu(Y_B^{sm}(a,b:c,d))=1+P_B^{25ps}(a,b)\mu(Y_A^{sm}(c+1,d))+(1-P_B^{25ps}(a,b))\mu(Y_A^{sm}(c,d+1))$

Recurrence Formulas

$\sigma^2(Y_A^{sm}(a,b))=p_A^{25s}\sigma^2(Y_B^{sm}(c+1,d))+q_A^{25s}\sigma^2(Y_B^{sm}(c,d+1))+p_A^{25s}q_A^{25s}(\mu(Y_B^{sm}(c+1,d))-\mu(Y_B^{sm}(c,d+1)))^2$

$\sigma^2(Y_B^{sm}(a,b))=q_B^{25s}\sigma^2(Y_A^{sm}(c+1,d))+p_B^{25s}\sigma^2(Y_A^{sm}(c,d+1))+q_B^{25s}p_B^{25s}(\mu(Y_A^{sm}(c+1,d))-\mu(Y_A^{sm}(c,d+1)))^2$

Boundary Values
$\sigma^2(Y_A^{sm}(a,b))= \sigma^2(Y_B^{sm}(a,b))=0$, if (c,d)=(3,0),(3,1),(0,3),(1,3)
$\sigma^2(Y_A^{sm}(2,2))= \sigma^2(Y_B^{sm}(2,2))=0$

$\sigma^2(Y_A^{sm}(a,b:c,d))=P_A^{25ps}(a,b)\sigma^2(Y_B^{sm}(c+1,d))+(1-P_A^{25ps}(a,b))\sigma^2(Y_B^{sm}(c,d+1))+P_A^{25ps}(a,b)(1-P_A^{25ps}(a,b))(\mu(Y_B^{sm}(c+1,d))-\mu(Y_B^{sm}(c,d+1)))^2$
$\sigma^2(Y_B^{sm}(a,b:c,d))=P_B^{25ps}(a,b)\sigma^2(Y_A^{sm}(c+1,d))+(1-P_B^{25ps}(a,b))\sigma^2(Y_A^{sm}(c,d+1))+P_B^{25ps}(a,b)(1-P_B^{25ps}(a,b))(\mu(Y_A^{sm}(c+1,d))-\mu(Y_A^{sm}(c,d+1)))^2$

Table 4 represents the mean and standard deviation of the number sets remaining in a match for different values of p_A and p_B.

	Mean		Standard deviation	
p_A, p_B	A serving	B serving	A serving	B serving
0.30, 0.30	4.1	4.1	0.8	0.8
0.30, 0.29	4.1	4.1	0.8	0.8
0.30, 0.25	4.0	4.0	0.8	0.8
0.25, 0.25	4.1	4.1	0.8	0.8
0.25, 0.24	4.1	4.1	0.8	0.8
0.20, 0.20	4.1	4.1	0.8	0.8

Table 4: The mean and standard deviation of the number sets in a match for different values of p_A and p_B

4. References

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Chapter 4: Australian Rules Football

1. Scoring Systems

In sporting events involving a fixed time duration of play, there is always the possibly of the result ending in a draw. Where a win must be obtained by a team (such as a final, teams advancing to the next round or in qualifying matches), different methods of breaking the tie are adopted. In soccer, if scores are level after extra time, the well-known penalty shoot-out is used to obtain a winner. Often extra time is given in sporting events to break the tie, and extra time may be given indefinitely until a winner is obtained. In finals matches (excluding the grand final) of Australian Rules football (AFL), an extra 10 minutes is given if teams are level after the fixed 80 minutes of playing time. If teams are level after the extra 10 minutes, then this process repeats until a winner is identified. In the grand final match of AFL, no extra time is given, and the entire match is played again the following week if teams are level after the fixed 80 minutes of play. If teams are level after the replayed match, then extra time will be played as given above for finals matches. In regular season round matches of AFL, no extra time is given if teams are level after the fixed 80 minutes of play, and the match is declared a draw with both teams being awarded 2 points each (a win in a round AFL match results in 4 points to the winner only). Note that AFL matches average 116 minutes of total time since the clock is stopped when the ball is not in play i.e. going out of the boundary of play. The model and analysis to follow will use a fixed 80 minutes of play and not consider stoppages.

In AFL 6 points are awarded for scoring a goal and 1 point is awarded for scoring a behind.

The distribution of margins and the mean and variance of margins are now obtained for round matches in AFL using recursive formulas. The probabilities of winning for Team A and Team B can be obtained from the distribution of margins. The probabilities of winning finals and grand finals matches are then obtained.

The following table represents scoring systems for rugby union, rugby league, soccer and AFL.

Sport	Duration (mins)	Scoring Shot	Points	Scoring Shot	Points	Scoring Shot	Points	Scoring Shot	Points
Rugby Union	80	Converted Try	7	Try	5	Penalty Goal	3	Field Goal	3
Rugby League	80	Converted Try	6	Try	4	Penalty Goal	2	Field Goal	1
Soccer	90	Goal	1						
AFL	80	Goal	6	Behind	1				

2. Chances of Winning

2.1 Notation for winning a match

Let p_{A6} and p_{B6} be the probabilities of Team A and Team B scoring a goal in a 1-minute time period respectively.
Let p_{A1} and p_{B1} be the probabilities of Team A and Team B scoring a behind in a 1-minute time period respectively.
Let p_{ns} be the probability of no score in a 1-minute time period.

Let us now define our assumptions:

1) $p_{A6}+ p_{B6}+ p_{A1}+ p_{B1} + p_{ns} = 1$
2) Independence holds for each time interval such that only one scoring shot (by either Team A or Team B), or no scoring shot can occur in a 1-minute time period.
3) Time is bounded by $0 \leq T \leq 80$
4) Margin is bounded by $-480 \leq d \leq 480$, since a team scoring 6 points (one goal) every minute for 80 minutes is 480 points.

Let $P_1(d,T)$ be the probability of Team A reaching a margin of d points ahead of Team B with T minutes remaining in the match.

Let w_{AR} and w_{BR} be the probabilities of Team A and Team B winning a round match (between Team A and Team B), and w_{DR} be the probability of a drawn match.

Let w_{A10} and w_{B10} represent the probabilities of Team A and Team B being ahead after 10 minutes of play and w_{D10} represent the probability that the scores are equal after 10 minutes of play.

Let w_{AF} and w_{BF} be the probabilities of Team A and Team B winning other finals matches (between them).

Let $P_4(T_1)$ represent the probability of other finals matches going for T_1 minutes in duration given teams are level after 80 minutes of play.

Let w_{AG} and w_{BG} be the probabilities of Team A and Team B winning a grand final match (between them).

2.2 Winning a match

Recurrence Formula
$P_1(d,T) = p_{A6}P_1(d-6,T+1) + p_{B6}P_1(d+6,T+1) + p_{A1}P_1(d-1,T+1) + p_{B1}P_1(d+1,T+1) + p_{ns}P_1(d,T+1)$, if $-480 \le d \le 480$ and $0 \le T \le 80$

Boundary Value
$P_1(0,80) = 1$

The distribution of margins for the match is obtained from $P_1(d,0)$, for $-480 \le d \le 480$.

<u>Winning round matches</u>
$w_{AR} = \sum_{0<d\le480} P_1(d,0)$, $w_{BR} = \sum_{-480\le d<0} P_1(d,0)$ and $w_{DR} = P_1(0,0)$.

It follows that $w_{AR} + w_{BR} + w_{DR} = 1$.

$w_{A10} = \sum_{0<d\le480} P_1(d,70)$, $w_{B10} = \sum_{-480\le d<0} P_1(d,70)$ and $w_{D10} = P_1(0,70)$

<u>Winning other finals matches</u>
$w_{AF} = w_{AR} + w_{DR}w_{A10}/(1-w_{D10})$, and $w_{BF} = 1- w_{AF}$.

<u>Winning grand final matches</u>
$w_{AG} = w_{AR} + w_{DR}w_{AF}$, and $w_{BG} = 1- w_{AG}$

Table 1 gives the probabilities of teams winning and drawing matches for all three scoring systems. The parameters used are based on averages from past AFL matches.

(1) p_{A6} =0.177, p_{B6}= 0.177, p_{A1}=0.156, p_{B1} = 0.156, p_{ns} = 0.334, and
(2) p_{A6} =0.213, p_{B6}= 0.141, p_{A1}=0.168, p_{B1} = 0.143 p_{ns} = 0.335.

	Rounds			Other Finals		Grand Final	
Parameters	w_{AR}	w_{BR}	w_{DR}	w_{AF}	w_{BF}	w_{AG}	w_{BG}
(1)	0.494	0.494	0.012	0.500	0.500	0.500	0.500
(2)	0.870	0.124	0.006	0.874	0.126	0.875	0.125

Table 1: The probabilities of teams winning and drawing matches for all of the currently used AFL scoring systems for two sets of parameter values

The results from Table 1 show that if both teams are evenly matched, then on average 1 in every 81 matches will end in a draw after 80 minutes of play. However, if one team is superior to the other, then on average 1 in every 154 matches will end in a draw. Therefore, it can be expected that 1 to 2 round matches in a season will result in a draw.

In the grand final match, the entire match is played again the following week if teams are level after the fixed 80 minutes of play. If teams are level after the replayed match, then extra time will be played as given for other finals matches. Therefore, the probability of a grand final match lasting for 160 minutes is obtained as w_{DR}. This is given by 0.012 for teams equal in ability and 0.006 when one team is superior over the other. These values are significantly greater than the probability of playing for 160 minutes in others finals matches, as reflected from table 2. From table 1, the probability of the superior team winning other finals matches (0.874) is about the same as that of winning a grand final match (0.875), yet the overall match duration is less for other finals matches. It could also be argued that replaying matches on the following week (if a draw is obtained) may cause complaints from spectators, given that spectators have paid for an event that has not been completed. Therefore, it could be argued that the system used in other finals matches is better than the system used in grand final matches since, although it has a slightly smaller probability of correctly identifying the better team, the expected period of time is reduced, and is possibly also better for spectator interest.

Table 2 gives the probabilities of other finals matches going for T_1 minutes in duration given teams are level after 80 minutes of play. These values can be used as a guide to determine whether the scoring system used in other finals matches could be applied to round matches to eliminate any possibility of a draw.

Parameters	20 mins	30 mins	40 mins
(1)	0.04713	0.00222	0.00010
(2)	0.04358	0.00190	0.00008

Table 2: The probabilities of other finals matches match going for T_1 minutes in duration given teams are level after 80 minutes of play

3. Margins

3.1 Notation for margins in a match

Let Y(T) represent a random variable of the number of margins remaining at time T minutes remaining in the match

Let $E(Y(T)) = \sum_{-480 \le d < 480} P_1(d,T)d$

Let $E(Y^2(T)) = \sum_{-480 \le d < 480} P_1(d,T)^2 d$

Let μ(Y(T)) represent the mean number of margins remaining at time T minutes remaining in the match

Let $\sigma^2(Y(T))$ represent the variance of the number of margins remaining at time T minutes remaining in the match

3.2 Mean and variance of margins remaining in the match

$\mu(Y(T)) = E(Y(T))$
$\sigma^2(Y(T)) = E(Y^2(T)) - E(Y(T))^2$

Table 3 represents the mean and standard deviation of margins remaining in the match from the outset for rounds matches.

	Rounds	
Parameters	μ(Y(0))	σ(Y(0))
(1)	25.8	19.5
(2)	40.4	26.6

Table 3: The mean and standard deviation of margins remaining in the match from the outset for rounds matches given parameters (1) and (2)

4. References

Barnett T (2011). Devising new Australian Rules football scoring systems. Journal of Quantitative Analysis in Sports 7(3)