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Obtaining a fair arbitration outcome

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The ‘minimax theorem’ is the most recognized theorem for determining strategies in a two-person zero-sum game. Other common strategies exist such as the ‘maximax principle’ and ‘minimize the maximum regret principle’. All these strategies follow the Von Neumann and Morgenstern linearity axiom which states that numbers in the game matrix must be cardinal utilities and can be transformed by any positive linear function $f(x) = ax + b$, $a > 0$, without changing the information they convey. This paper describes risk-averse strategies for a two-person zero-sum game where the linearity axiom may not hold. With connections to gambling theory, there is evidence to show why it can be optimal for the favourable player to adopt risk-averse strategies. Based on this approach, an arbitration value is obtained in a litigation game, where the amount awarded to the victim is less than expectation and shown to be ‘fairer’ when compared with the amount obtained using the Von Neumann and Morgenstern game theory framework.

Keywords: arbitration; risk-averse; game theory; dispute resolution; lawsuits.

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1. Introduction

The origins of game theory extend back to 0–500 AD where the so called marriage contract problem is discussed in the Talmud. In 1713, James Waldegrave provided the first known minimax mixed strategy solution to a two-person game, but expressed concern that a mixed strategy ‘does not seem to be in the usual rules of play’ of games of chance. Perhaps, the ‘official’ beginning of game theory was the 1944 book ‘Theory of Games and Economic Behavior’ (Von Neumann and Morgenstern, 1944). Two-person zero-sum game theory is covered as well as the framework for modern axiomatic utility theory by assigning numbers to outcomes in a way that reflect an actor’s preference. It was also the account of axiomatic utility theory given there that led to its widespread adoption within economics and law. One of the axioms states that numbers in the game matrix must be cardinal utilities and can be transformed by any positive linear function $f(x) = ax + b$, $a > 0$, without changing the information they convey.

Consider Game 1 and Game 2, where Game 2 is a linear transformation of Game 1 by adding $b = 4$ to all utility values given by Game 1. The strategies given by the minimax theorem are P1: $A = 0.2$, $B = 0.8$ and P2: $A = 0.7$, $B = 0.3$. P1 may choose to always play Strategy B in Game 1 to guarantee at least a positive payout of +2. If P1 was to use the strategy as given by the minimax theorem in Game 1, then he could end up with a negative payout of -3, even though the expected payout of 2.6 is positive. However, P1 may choose to play the strategies given by the minimax

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theorem in Game 2 since it is always guaranteed a positive payout of +1. This example contradicts the Von Neumann–Morgenstern linearity axiom and indicates that while a player may choose to maximize expectations, a player (presumably the player with an expected positive payout) may be concerned about obtaining a negative payout or their maximum possible loss (MPL) for the game. This is likely to be more of a concern when the MPL is a negative payout and to reduce the MPL occurring, a player may choose strategies accordingly. In effect, the maximal expectation is reduced by minimizing the probability of obtaining the MPL, i.e. using risk-averse strategies. Risk-averse strategies have foundations in gambling theory when a favourable game exists. Friedman (1980) and Schlesinger (2004) outline risk-averse strategies in blackjack where the objective is to minimize the probability of losing one's bankroll.

		P2	
		A	*B*
P1	*A*	5	−3
	B	2	4

Game 1

		P2	
		A	*B*
P1	*A*	9	1
	B	6	8

Game 2

Game Theory and the Law (Baird *et al.*, 1994) is one of the first books in law and economics to take an explicitly game theoretic approach to the subject. The following model is used in the litigation process. Injurer's car ran over victim, and victim suffered \$100 000 in damages. Litigation would cost each party \$10 000. It is estimated that the probability of victim prevailing in court is 75%. The injurer should prefer settling rather than litigating at any amount <\$85 000 and the victim should prefer settling rather than litigating at any amount >\$65 000. Therefore, the amount of settlement between the injurer and the victim should be between \$65 000 and \$85 000. Note that this model does not take into account that the favourable player (victim) may be risk-averse by reducing the 25% chance of being unsuccessful in litigation (and losing \$10 000 in the process) and therefore would be willing to accept <\$65 000 in an out-of-court settlement. Suppose arbitration was used to determine the amount that the injurer was to pay to the victim. Based on the model developed in Baird *et al.* (1994), a fair arbitration amount could be \$75 000, as the victim has a 75% chance of receiving \$100 000 from the injurer if the dispute went to litigation. However, based on the reasoning above, the arbitration amount to the victim could possibly be less than the expectation of \$75 000 since the victim has a 25% chance of losing \$10 000 in litigation, even though the expected payout in litigation is positive. This article will address this observation by firstly analyzing two-person zero-sum games in table form and apply the results to the form given in litigation games.

2. Existing strategies

2.1 *Minimax theorem*

Consider the following 2 × 2 game (call it Game 3)

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		P2	
		A	*B*
P1	*A*	2/3	−1
	B	−1/3	1

Game 3

The most common solution to this type of game is using the minimax theorem to obtain player strategies. This gives mixed strategies as P1: 4/9*A*, 5/9*B* and P2: 2/3*A*, 1/3*B*. The value of the game is 1/9. Table 1 represents Game 3 in a form where both players apply strategies from the minimax theorem. Outcome *AA* refers to P1 using Strategy *A* and P2 using Strategy *A*. Outcomes *AB*, *BA* and *BB* are obtained similarly. The value of the game is given as 1/9 as expected. This representation is typically given for a casino game.

Table 2 is an extension of Table 1 which is used to obtain moments, which can then be used to obtain cumulants and common distributional characteristics as follows:
Mean = 0.111; standard deviation = 0.703; coefficient of variation = 6.325; coefficient of skewness = −0.158; and coefficient of excess kurtosis = −1.425.

These distributional characteristics (other than the mean) provide extra information to the possible payouts in a two-person zero-sum game.

2.2 *Other strategies*

Commonly used strategies other than the strategies determined by the minimax theorem are as follows as documented in Straffin (1993): from Game 3, suppose P1 recognizes that P2 will play mixed strategies as 0.8*A*, 0.2*B*. Then using the expected value principle, P1 would use the fixed strategy of *A* since the reward for the game is 1/3. Players may want to be cautious and Wald's method for P1 is to write down the minimum entry in each row and choose the row with the largest minimum. This would mean P1 would use a fixed strategy of *B*. An analog for P2 would be to use the fixed strategy of *A*. As Wald's 'maximin' strategy is looking at the worst that will happen, the corresponding

TABLE 1 *Probabilities and expected payouts for Game 3 with both players using strategies under the minimax theorem*

Outcome	Payout	Probability	Expected payout
AA	2/3	4/9 × 2/3 = 8/27	2/3 × 8/27 = 16/81
AB	−1	4/9 × 1/3 = 4/27	−1 × 4/27 = −4/27
BA	−1/3	5/9 × 2/3 = 10/27	−1/3 × 10/27 = −10/81
BB	1	5/9 × 1/3 = 5/27	1 × 5/27 = 5/27
		1	1/9

TABLE 2 *The first four moments for Game 3 with both players using strategies under the minimax theorem*

Outcome	Payout	Probability	First moment	Seccond moment	Third moment	Fourth moment
AA	2/3	8/27	0.198	0.132	0.088	0.059
AB	−1	4/27	−0.148	0.148	−0.148	0.148
BA	−1/3	10/27	−0.123	0.041	−0.014	0.005
BB	1	5/27	0.185	0.185	0.185	0.185
			0.111	0.506	0.111	0.396

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principle for optimists would be the 'maximax' principle. This would mean P1 would use a fixed strategy of *B* and P2 would use a fixed strategy of *B*. Hurwicz combined these two approaches by choosing a 'coefficient of optimism' α between 0 and 1. For each row, compute α (row maximum) + $(1-\alpha)$(row minimum). Choose the row for which this weighted average is the highest. For example, suppose we choose $\alpha = 0.8$, then *A*: 0.8 (2/3) + 0.2 (−1) = 1/3 and *B*: 0.8 (−1/3) + 0.2 (1) = −1/15. Hence, P1 would choose Strategy *A*. Savage's method involves regret by writing down the largest entry in each row. Choose the row for which this largest entry is smallest. This would mean P1 would choose the fixed strategy of *A* and P2 would choose the fixed strategy of *A*.

As mentioned Section 1, Von Neumann and Morgenstern (1944) developed the framework for modern utility theory by assigning numbers to outcomes in a way that reflect an actor's preference. It is stated that for a mixed strategy game solution to be meaningful, the numbers in the game matrix must be cardinal utilities and can be transformed by any positive linear function $f(x) = ax + b, a > 0$, without changing the information they convey. All the strategies in Section 2.2 and the strategies determined by the minimax theorem conform to this property. However, this property does not take into account risk and whether a player (presumably the player with an expected positive payout) would reduce the expected payout in order to reduce risk; such as minimize the probability of obtaining the maximum possible loss. Section 3 will outline methods to reduce risk for the favourable player in a two-person zero-sum game.

3. Risk-averse strategies

3.1 *Distributional characteristics*

		P2	
		A	*B*
P1	*A*	*a*	*b*
	B	*c*	*d*

Game 4

Consider a general 2 × 2 zero-sum game as given by Game 4. To apply mixed strategies using the minimax theorem, we will assign $d > a > c > b$ and $b < 0$. This gives the optimal mixed strategy for P1 as $(d-c)/[(d-c)+(a-b)]A$, $(a-b)/[(d-c)+(a-b)]B$. The value of the game is calculated as $\upsilon = (ad-bc)/[(a-c)+(d-b)]$. Suppose P1 is the favourable player such that $\upsilon > 0$. P1 can deviate from minimax strategies by either increasing Strategy *A* (which decreases Strategy *B*) or increasing Strategy *B* (which decreases Strategy *A*). By increasing Strategy *A*, the MPL consisting of payout *b* will increase and similarly by increasing Strategy *B*, the MPL will decrease. Therefore, the deviation from minimax strategies for P1 can be thought of in terms of the increase or decrease in the MPL. Since P1 can guarantee an expected positive payout by playing minimax strategies, we will assume that any strategy that P1 adopts will give an expected positive payout regardless of the strategies used by P2.

Table 3 represents the distributional characteristics for Game 3 with P2 using minimax strategies in all columns and P1 using minimax strategies (column 2), strategies that decrease the MPL (column 3) and strategies that increase the MPL (column 4). It is observed from column 3 that the standard deviation, coefficient of variation and coefficients of skewness and excess kurtosis are reduced when compared with column 2. It is observed from column 4 that the standard deviation, coefficient

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TABLE 3 *Distributional characteristics of Game 3 for various strategies used by P1*

Distributional characteristic	P1: 4/9A, 5/9B P2: 2/3A, 1/3B	P1: 0.4A, 0.6B P2: 2/3A, 1/3B	P1: 0.48A, 0.52B P2: 2/3A, 1/3B
Mean	0.111	0.111	0.111
Standard deviation	0.703	0.696	0.708
Coefficient of variation	6.325	6.261	6.375
Coefficient of skewness	−0.158	−0.095	−0.206
Coefficient of excess kurtosis	−1.425	−1.424	−1.427
MPL	0.148	0.133	0.160
Overall loss	0.519	0.533	0.507

of variation and coefficients of skewness and excess kurtosis are increased when compared with column 2.

3.2 *Definition for a* 2 × 2 *zero-sum game*

Based on the results from Section 3.1, Definition 1 describes risk-averse strategies for a 2 × 2 zero-sum game.

DEFINITION 1. Consider a 2 × 2 zero-sum game where at least one of the payouts is positive and at least one of the payouts is negative. The value υ of the game under the minimax theorem is either positive, negative or zero. Risk-averse strategies can be obtained when v is positive such that the expected payout for P1 is positive regardless of the strategies used by P2, and the probability of obtaining the MPL for P1 in the game is reduced when compared with the strategies under the minimax theorem. Risk-averse strategies for P2 when υ is negative follow.

Using Definition 1, the risk-averse solution for P1 to Game 3 is obtained as P1: $1/3 < A < 4/9$, $5/9 < B < 2/3$. These calculations were obtained by noting that P1 always obtains an expected positive payout regardless of the strategies used by P2.

EXAMPLE 1. Using Game 3, suppose P1 is restricting the probability of the −1 payout to be 0.12 (given P2 uses mixed strategies obtained from the minimax theorem). Then $0.12/1/3 = 0.36$ and P1 should use the risk-averse strategy of P1: 0.36A, 0.64B. If P2 identified that P1 was deviating from the minimax theorem, then P2 could use Strategy A for an expected payout for P2 of −0.027.

3.3 *Risk of ruin*

A common problem that often arises in gambling is obtaining the probability of losing one's entire bankroll given a favourable game. The following recursive solution (which assumes independent trials) was derived by Evgeny Sorokin and posted on Arnold Snyder's Blackjack Forum Online http://www.blackjackforumonline.com/content/VPRoR.htm.

The equation is given as $R(1) = E[p_i \times R(1)^{Z_i}]$, where $R(1)$ is the risk of losing a 1-unit bankroll, Z_i is the return payoff for outcome i and p_i is the associated probability for Z_i.

In the context of Game 3, suppose P1 has a bankroll of 3 units. With P1 and P2 playing strategies under the minimax theorem, the risk of ruin for P1 is obtained as 26.0877%. Suppose P1 increased Strategy B to 0.613, then the risk of ruin (with P2 playing strategies under the minimax theorem) is

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reduced to 24.87227%. If P2 then increased Strategy A (to reduce the expected payout for P1), the risk of ruin for P1 would only decrease as shown in Table 4.

Therefore, the risk of ruin is reduced by P1 playing Strategy B with probability 0.613 regardless of the strategies used by P2.

3.4 *Kelly criterion*

The Kelly criterion (Kelly, 1956) is typically applied to favourable casino games to maximize the long-term growth of the bankroll. The Kelly criterion for when multiple outcomes exist is given as follows (Barnett, 2010).

Consider a game with m possible discrete finite mixed outcomes. Suppose the profit for a unit wager for outcome i is k_i with probability p_i for $1 \leqslant i \leqslant m$, where at least one outcome is negative and at least one outcome is positive. Then if a winning strategy exists, and the maximum growth of the bank is attained when the proportion of the bank bet at each turn, b, is the smallest positive root of $\sum_{i=1}^{m} \frac{k_i p_i}{1+k_i b} = 0$.

The Kelly criterion is applied to Game 3 to demonstrate why the favourable player may consider risk-averse strategies. The value of b with both P1 and P2 using minimax strategies is obtained as 0.222. Therefore, P1 (the favourable player) should wager an amount of $0.222 \times$ current bankroll to maximize the long-term growth of the bank. Since the amount that P1 can bet is fixed at 1 unit, the decision on whether to apply minimax strategies or risk-averse strategies can depend on the size of the bankroll. Using Solver in Excel, Table 5 shows that P1 would need a bankroll of at least 4.51 units to avoid over betting by using minimax strategies. By using risk-averse strategies, P1's bankroll can be <4.51 as given in Table 5.

3.5 *Definition for an $m \times n$ zero-sum game*

In a 2×2 zero-sum game, risk-averse strategies were obtained for the favourable player by reducing the probability of the MPL occurring. This idea of reducing the MPL is extended to $m \times n$ zero-sum games.

TABLE 4 *Risk of ruin and expected payouts for Game 3 for various strategies used by P1 and P2*

	P1: $A = 0.387$, $B = 0.613$	
Probability	Risk of ruin for P1 (%)	Expected payout
P2: $A = 2/3$	24.87227	0.11111
P2: $A = 0.667$	24.87197	0.11105
P2: $A = 0.668$	24.87107	0.11088
P2: $A = 0.70$	24.84051	0.10537

TABLE 5 *Bankroll requirements when using strategies under the Kelly criterion*

Strategy	Bankroll (units)
P1: $A = 4/9$, $B = 5/9$	4.51
P1: $A = 0.4$, $B = 0.6$	4.38
P1: $A = 0.35$, $B = 0.65$	4.23

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Consider an $m \times n$ zero-sum game where at least one of the payouts is positive and at least one of the payouts is negative. The value υ of the game under the minimax theorem is either positive, negative or zero. Risk-averse strategies can be obtained when v is positive such that the expected payout for P1 is positive regardless of the strategies used by P2, and the probability of obtaining the MPL for P1 in the game is reduced when compared with the strategies under the minimax theorem. Risk-averse strategies for P2 when υ is negative follow.

4. Arbitration for a two-person zero-sum game

4.1 *Table form*

Suppose the payouts for each player in Game 3 was to be determined by an outside arbitrator. One obvious method is simply to use the value of the game given by the minimax theorem. For Game 3, this would be 1/9 to P1. However, it would appear to be more of an incentive for the favourable player to have the game determined by arbitration rather than play the game simultaneously, as they run the risk of being at a loss even though the expected outcome is positive. For example, from Table 3, if both players are playing minimax strategies, there is a 0.148 probability of ending up with the MPL of -1 on any trial and a 0.519 probability of ending up with any loss on any trial. The favourable player can of course reduce the MPL by playing risk-averse strategies but as a consequence could reduce the expected amount if the other player changed strategies accordingly. This illustration suggests that the arbitration amount to the favourable player should be less than the expected amount as given by the minimax theorem.

Suppose the game given by Table 1 was a casino game and the player had a finite bank. If the player bet the same amount on each trial, then the expected profit on each trial would be 0.11111 for a unit bet. If the player had a bankroll of 3 units, then the chance of ruin as given in Section 3.3 is 26.0877%. The expected profit on each trial differs under the Kelly criterion method since the player's bankroll changes each trial according to the wins and losses, and hence we will adopt an averaged expected profit notation. If the player applied the Kelly criterion with a bankroll of 3 units, then the averaged expected profit on each trial would be $0.11111 \times 0.2271 = 0.0246$, and the chance of ruin would approach zero. Given the Kelly criterion maximizes the long-term growth of the bank, this would appear to be a reasonable method in a favourable gambling context and shows that the averaged expected amount of profit is less than the amount given by fixed betting on each trial. Based on this reasoning, an arbitration value could be determined by the averaged expected profit as given under the Kelly criterion. For Game 3, this value is given as 0.0246.

4.2 *Litigation form*

As given in Section 1, Baird *et al.* (1994) use the following model in the litigation process. Injurer's car ran over victim and victim suffered \$100 000 in damages. Litigation would cost each party \$10 000. It is estimated that the probability of victim prevailing in court is 75%. The injurer should prefer settling rather than litigating at any amount $<$\$85 000 and the victim should prefer settling rather than litigating at any amount $>$\$65 000. Therefore, the amount of settlement between the injurer and the victim should be between \$65 000 and \$85 000. Note that this model does not take into account that the favourable player (victim) may be risk-averse by reducing the 25% chance of being unsuccessful in litigation (and losing \$10 000 in the process) and therefore would be willing to accept $<$\$65 000 in an out-of-court settlement. As shown in Section 3.4, risk-averse strategies

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can be optimal with the objective of maximizing the long-term growth of the bank, even though the expectation is reduced.

Barnett (2010) applied the Kelly criterion to lawsuits to obtain insights in the decision-making process as to whether it is beneficial for a victim to file a lawsuit against the injurer. The analysis can be used to determine whether a victim should have legal representation in court to obtain a higher expected payout or minimize risk through legal costs by representing themselves in court, even though the expected payout is reduced without legal representation. Analysis was also given to obtain insights as to how much a victim should accept in an out-of-court settlement. Applying the model from Barnett (2010) to the example above, the victim should prefer settling rather than litigating at any amount $>\$65\,000 \times b = \$65\,000 \times 0.722 = \$46\,944$, where b is the Kelly betting fraction from Section 3.4. Therefore, the amount of settlement between the injurer and victim should be between \$46 944 and \$85 000.

Suppose arbitration was used to determine the amount that the injurer was to pay to the victim. Using the model developed in Baird *et al.* (1994), a fair arbitration amount could be \$75 000, as the victim has a 75% chance of receiving \$100 000 from the injurer if the dispute went to litigation. Using the model developed in Barnett (2010) based on the Kelly criterion, a fair arbitration amount could be $(\$65\,000 \times 0.722) + \$10\,000 = \$56\,944$, as \$46 944 is the averaged expected profit under the Kelly criterion that the victim is most likely to obtain if the case went to litigation with the addition of the \$10 000 in legal costs. Note that this arbitration amount awarded to the victim is 24% less than the amount given by the model developed in Baird *et al.* (1994).

A general arbitration formula V is given as:

$$V = E * b + C,$$

where E is the expected payout, b is the Kelly betting fraction, C is 50% of the total legal costs between the two parties.

5. Conclusions

The game theory framework is built around the Von Neumann and Morgenstern axiomatic utility theory. This paper has devised risk-averse strategies for the favourable player in a two-person zero-sum game where the linearity axiom may not hold. Using the Kelly criterion (as typically used in favourable casino games), an arbitration value is obtained in a litigation game based on the observation that the victim may end up with a negative payout if the dispute went to litigation even though the expected payout is positive. This arbitration amount to the victim is less than expectation and shown to be fairer when compared with the amount obtained using the Von Neumann and Morgenstern game theory framework.

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