GAMING LAW REVIEW AND ECONOMICS
Volume 14, Number 8, 2010

DOI: 10.1089/glre.2010.14806

Applying Mathematics to Poker Machine Regulations to Increase Consumer Protection

Tristan Barnett

APPLYING MATHEMATICS TO CASINO GAMES

Casino games are comprised of mathematical formulations which can be found readily in the literature, such as Croucher,[1] Hannum,[2] Packel,[3] and Epstein.[4] The percent house margin (or return to player) establishes how much a player is expected to lose in the long run. While the percent house margin is important to consumers (players are consumers of casino games) in determining the choice of games or how long to play a particular game, there is other information which could also influence these decisions.

For example, the probability of the consumer ending up in profit after 100 games, or the probability of the consumer losing more than $100 after 200 games, would be valuable information. These results are classified as the distribution of payouts, and to calculate these results requires three pieces of information from the casino game: 1.) the initial cost; 2.) the payouts for each possible outcome; and 3.) the probability associated with each outcome. The initial cost (the cost to play) is given directly for any casino game. The payouts for each possible outcome are also given directly for every casino game, either in the form of odds or prices.

On the other hand, the probabilities associated with each outcome are not given directly for a casino game. However, these probabilities can usually be derived from the playing rules. For example, in single zero roulette, the probability of a particular number coming up can easily be calculated as 1/37, since there is one favorable outcome out of 37 possible outcomes. It can therefore be argued that games where the probabilities associated with each outcome can be obtained from the playing rules are "fair," since the distribution of payouts can be readily obtained using mathematics. On the contrary, casino games where the probabilities associated with each outcome cannot be obtained from the playing rules could be considered as "unfair," since important information is hidden from the player. This is the situation for poker machines in Australia.

Dr. Tristan Barnett is managing director for Strategic Games and adjunct lecturer, School of Management and Information Systems, at Victoria University in Melbourne, Australia.

PERCENT HOUSE MARGIN

A casino game can be defined as follows:

- There is an initial cost C to play the game.
- With the assumption of trials being independent,[5] each trial results in an outcome O_i, where each outcome occurs with profit x_i and probability p_i.
- The condition $\Sigma p_i = 1$ must be satisfied, which means that sum of all the probabilities must equal 1.

[1] J.S. Croucher, The Statistics of Gambling (2002).
[2] Robert C. Hannum and Anthony N. Cabot, Practical Casino Math (2d ed. 2005).
[3] EDWARD PACKEL, The Mathematics of Games and Gambling (1981).
[4] Richard A. Epstein, The Theory of Gambling and Statistical Logic (1997).
[5] If each trial is independent, then the outcome of one trial does not affect the outcome of any other trial. For example, when a coin is flipped, the chance of it coming up heads or tails does not depend on the outcome of the previous toss.

Given this information, $E_i = p_i x_i$ provides the expected profit E_i for each outcome,[6] and the total expected profit is given by ΣE_i.

The *percent house margin* (%HM) is then $-\Sigma E_i/C$ and the total return is $1 + \Sigma E_i/C$. Positive percent house margins indicate that the gambling site on average makes money and the players lose money. Table 1 summarizes this information. For interested readers, calculating *moments and cumulants* and the *distribution of profits* are detailed in the Appendix to this article.

POKER MACHINE REGULATIONS

The Australian/New Zealand Gaming Machine National Standard (Standard) has been developed by participants from various gaming regulators to outline gaming machine regulations common to all jurisdictions. The following is documented in the Standard Revision 9.0:

> The National Standard Working Party was established by the Australian and New Zealand gaming regulators on 21 March 1994. The purpose of the working party is to develop technical requirement documents to be used by each individual jurisdiction as the basis for working towards a common technical requirement for the evaluation of gaming machines. The intent of this document is to ensure gaming on gaming machines occurs in a manner that is: a) fair; b) secure; and c) auditable and that gaming machines are reliable in terms of these issues. It is anticipated that amendments to the Standard will occur on an annual basis and then be adopted by the participating jurisdictions. Individual jurisdictions can be expected to amend their respective requirements documents in advance of the National Standard where player fairness, security or auditability is considered to be jeopardized.

TABLE 1. REPRESENTATION OF A CASINO GAME

Outcome	*Profit*	*Probability*	*Expected Profit*
O_1	x_1	p_1	$E_1=p_1x_1$
O_2	x_2	p_2	$E_2=p_2x_2$
O_3	x_3	p_3	$E_3=p_3x_3$
...	...	...	...
		1	ΣE_i

DISPLAYING PROBABILITIES

A consumer's decision as to the choice of game or how long to play a particular game may be influenced by knowing the distribution of payouts. To calculate the distribution of payouts on a poker machine requires the probabilities associated with each particular payout. The probabilities on poker machines cannot be obtained from the playing rules (as is the case with table games), and therefore poker machines could be considered as being "unfair."

As an approach towards responsible gambling and to increase consumer protection, the probabilities associated with each payout could be displayed on the machine, along with information that would advise players of their probability of ending up with a certain amount of profit after N trials, or after playing a certain amount of time.

Table 2 represents the first four moments of a sample poker machine after one trial, where $x = 1/8000 + 1/800 + 1/80 + 1/8$. The percent house margin is calculated as 10%. By applying the Normal Power Approximation (detailed in the Appendix), Table 3 represents the probabilities of obtaining various profit intervals after 1,080 spins, playing \$1 hands. Assuming 18 spins per minute on a typical poker machine, a player is likely to spin $18 * 60 = 1{,}080$ spins per hour. (Table 3 therefore shows the probabilities of different profit intervals after an hour of play.) Note that even though the expected loss after 1,080 spins is −\$108, a player has a 35.5% chance of losing more than \$300. Section 3.9.9 in the Standard states that—

> A gaming machine must display the following information to the player:
>
> a) the player's current credit balance;
> b) the current bet amount;
> c) all possible winning outcomes, or be available as a menu item or help menu;
> d) win amounts for each possible winning outcome or be available as a menu or help screen item;

[6] The potential profit for an outcome in any game multiplied by the probability of that outcome determines the expected profit. If a person stands to make \$1 profit when a tossed coin comes up heads, then his expected profit is \$0.50, because \$1.00 times 0.5 (the coin will land heads 50% of the time) is \$0.50.

TABLE 2. THE FIRST FOUR MOMENTS OF A SAMPLE POKER MACHINE AFTER ONE TRIAL

Outcome	*Profit ($)*	*Probability*	*1st Moment*	*2nd Moment*	*3rd Moment*	*4th Moment*
O_1	1000	1/8000	0.125	125	125000	125000000
O_2	100	1/800	0.125	12.5	1250	125000
O_3	10	1/80	0.125	1.25	12.5	125
O_4	1	1/8	0.125	0.125	0.125	0.125
O_5	0	2/5 − x	0	0	0	0
O_6	−1	3/5	−0.6	0.6	−0.6	0.6
		1.000	−0.100	139.475	126262.025	125125125.725

e) the amount won for the last completed play (until the next play starts, or following player input related directly to the next play); and

f) the player options selected (e.g. bet amount, lines played) for the last completed play (until the next play starts, or following player input related directly to the next play).

Based on the above argument, clause d) could instead read: "d) win amounts for each possible winning outcome with associated probabilities or be available as a menu or help screen item."

WIN AMOUNTS

There are two common ways of displaying payouts on casino games: odds or prices. *Odds* are typically used in casino table games. Odds are written in the form of x to y, which means that if the player is successful they receive a return of (x/y) * initial cost + initial cost, or a profit of (x/y) * initial cost.[7] Note the difference between the return and the profit.

Prices are typically used in poker machines. Prices are written in the form of $z, which means that a successful player receives a return of $z * initial cost, or a profit of $z * initial cost – initial cost. Odds and prices can easily be interchanged.

TABLE 3. THE PROBABILITIES OF OBTAINING VARIOUS PAYOUTS AFTER 1,080 SPINS OF A SAMPLE POKER MACHINE

Profit interval ($)	*Probability*
<−300	0.355
−300 to −100	0.202
−100 to 0	0.099
0 to 100	0.089
>100	0.251

Poker machines, however, can be misleading since winning on a machine refers to returns rather than profits. For example, if a player bets $10 on a machine and receives a return of $7, then the player has actually lost $3, receiving a profit of −$3. Notwithstanding the loss, the machine indicates that the player has won $7. It can be misleading to players to use win amounts as the return rather than the profit.

An extra clause could be added to Section 3.9.9 in the Standard to address this: "g) win amounts refer to profits (rather than returns)."

WITHDRAWING CREDITS

Poker machines in Australian gaming venues only allow deposits and withdrawals directly to or from the machine in multiples of $1. Players may withdraw money when remaining credits are less than $1, but this requires calling over the attendant and having the remaining money paid by the attendant, rather than directly from the machine. This can be a relatively lengthy process, and may require formal identification and a signature by the player. A player may decide to gamble less than $1, to leave the venue after a certain period of time, or to cash out after a relatively big win. In these scenarios, the player may feel it is too much trouble to call over the attendant to receive amounts less than $1, and instead decide to either play out the remaining credit of less than $1, or simply leave the remaining credit in the machine. As a result, players will be often effectively be deprived of credits owed to them of less than $1.

As an alternative, by having machines have a withdrawal denomination of 5 cents (as well as $1),

[7] For example, if $2 is bet at 3 to 2 odds, the player recovers $5 if he wins, representing a return of his $2 wager plus $3 profit.

players would leave at most 4 cents "on the table." For example, with 99 cents remaining in a machine, a player could withdraw 95 cents (the remaining 4 cents would not be rounded to 5 cents and withdrawn). This withdrawal process could be included as a regulation in the Standard.

INITIAL COST

Poker machines vary in the initial cost depending on the denomination of the machine (1¢ to $1 are common) and the number of lines (1 to 10 are common). The initial cost that a player chooses is given by the number of lines multiplied by the denomination of the machine. For example, 5 lines on a $1 machine would amount to an initial cost of $5. The minimum initial cost that a player could play is 1¢, by playing 1 line on a 1¢ machine. Assuming 18 spins per minute and a percent house margin of 13%, someone playing at an initial cost of 1¢ would be expected to lose 0.13 * 18 * 60 * $0.01 = $1.40 per hour.

Playing 1 line on a 1¢ poker machine is one of the best games on offer for minimizing losses over a period of time, and could be used as an approach to responsible gambling. There is currently no regulation as to how the total number of gaming machines at each venue should be proportioned among different denominations. It is possible, for example, for a gaming venue to offer no machines with a 1¢ denomination. This could be considered unfair to players that want to gamble on poker machines for the purpose of minimizing losses. The Standard could include documentation as to how the total number of gaming machines at each venue should be proportioned by different denominations.

STANDARD DEVIATION

The mean profit is one of many distributional characteristics. Other distributional characteristics are the standard deviation and the coefficients of skewness and excess kurtosis. These four characteristics can be applied directly in the Normal Power Approximation formula to estimate the distribution of profits.

Consider the two games given in Table 4. The initial cost for Game 1 is $1, whereas the initial cost for Game 2 is $2. The %HM for both games is 10% and they are effectively the same game. However, the standard deviations for these two games are calculated as 0.995 for Game 1 and 1.990 for Game 2. These values are different because the standard deviation is dependent on the mean profits, which are −$0.10 for Game 1 and −$0.20 for Game 2. Since a greater standard deviation means more variation from the mean or average, these two games, despite having the same %HM, are not identical.

The Standard in section 3.9.17 has restrictions on the standard deviation and states: "The Nominal Standard Deviation (NSD) of a game must be no greater than 15. In determining the NSD for a game, the following conventions must be applied: a) Calculate standard deviation of the base game at minimum bet and single line play or equivalent."

A problem with this is that the base game could be an initial cost of $1, as in Game 1, or $2 as in Game 2. Increasing the initial cost increases the standard deviation; therefore, the standard deviation is not consistent for a particular poker machine. One way to solve this problem is to normalize all poker machine games to a fixed initial cost when performing calculations on the standard deviation, and then deciding on a reasonable NSD. Section 3.9.17 could be amended to reflect this argument.

COEFFICIENTS OF SKEWNESS AND EXCESS KURTOSIS

The Standard has restrictions on the standard deviation, such that the NSD of a game must be no greater than 15. The Standard in section 3.9.16b also

TABLE 4. A COMPARISON OF TWO GAMES WITH THE SAME %HM, WHERE THE INITIAL COST FOR GAME 1 IS $1 AND THE INITIAL COST FOR GAME 2 IS $2

	Game 1		*Game 2*	
Outcome	*Profit ($)*	*Probability*	*Profit ($)*	*Probability*
O_1	1	0.45	2	0.45
O_2	−1	0.55	−2	0.55

gives restrictions on the probabilities and states: "The probability for attaining each winning pattern of symbols (offered in the Base game) must not be less than 1/7,000,000."

Consider the game given in Table 5, where x = 1/7,000,000. The only way a player can profit is by obtaining a $39,000 profit with the unlikely probability of 1 in 7 million! The standard deviation is calculated as 14.745, which is less than the 15 restriction. None of the probabilities are less than 1/7,000,000, which is in agreement with the above statement. Therefore, the probabilities and payouts for this game meet the regulations given in the Standard.

One way to prevent such a game from occurring is to provide restrictions in the Standard for the maximum amounts for two other distributional characteristics, the coefficients of skewness and excess kurtosis. For the game given in Table 5, the coefficient of skewness is calculated as 2,643.553 and the coefficient of excess kurtosis is calculated as 6,992,243.972. For comparison, the standard deviation and coefficients of skewness and excess kurtosis from the game given in Table 2 are 11.810, 76.686, and 6,432.605, respectively. Setting limits on the coefficients of skewness and excess kurtosis will prevent games such as those seen in Table 5, above.

SIX SIMPLE WAYS TO INCREASE CONSUMER PROTECTION

Mathematical and logical reasoning suggest several straightforward amendments to the Standard for poker machines, with the purpose of increasing consumer protection. These possible amendments are:

1. The probabilities associated with the payouts should be displayed on the gaming machine.
2. Win amounts should refer to profit payouts rather than return payouts.
3. Gaming machines should allow players to withdraw amounts less than $1.
4. The total number of gaming machines at each venue should be proportioned by different denominations.
5. The standard deviation should be regulated on gaming machines with a fixed initial cost that is consistent across all machines.
6. There should be regulations for the coefficients of skewness and excess kurtosis.

APPENDIX

Moments and cumulants

The outcome or profit from a single bet, X, is a random variable. From probability theory, the moment generating function (MGF) of X is

$$M_X(t) = E(\exp(Xt))$$

$$= 1 + m_{1X}t + m_{2X}t^2/2! + m_{3X}t^3/3! + m_{4X}t^4/4![+t] \dots$$

where m_rX represent the r^{th} moment of X. The moments of X are readily calculated using

$$m_{1X} = \Sigma_i\ p_i x_i$$

$$m_{2X} = \Sigma_i\ p_i x_i^2$$

$$m_{3X} = \Sigma_i\ p_i x_i^3$$

$$m_{4X} = \Sigma_i\ p_i x_i^4$$

and so on. The calculation of these moments is illustrated in Table 6.

The cumulant generating function (CGF) of X is the natural log of the MGF:

$$K_X(t) = \log_e(M_X(t)$$

TABLE 5. THE FIRST FOUR MOMENTS OF A CASINO GAME AFTER ONE TRIAL

Outcome	*Profit ($)*	*Probability*	*1st Moment*	*2nd Moment*	*3rd Moment*	*4th Moment*
O_1	39,000	1/7,000,000	0.006	217.286	8474142.857	330491571428.571
O_2	0	0.86 − x	0	0	0	0
O_3	−1	0.14	−0.140	0.140	−0.140	0.140
		1.00	−0.134	217.426	8474142.717	330491571428.711

TABLE 6. REPRESENTATION OF THE FIRST FOUR MOMENTS OF THE PROFIT OF A CASINO GAME AFTER ONE BET

Outcome	*Profit*	*Probability*	*1st Moment*	*2nd Moment*	*3rd Moment*	*4th Moment*
O_1	x_1	p_1	p_{1x1}	p_{1x1}^2	p_{1x1}^3	p_{1x1}^4
O_2	x_2	p_2	p_{2x2}	p_{2x2}^2	p_{2x2}^3	p_{2x2}^4
O_3	x_3	p_3	p_{3x3}	p_{3x3}^2	p_{3x3}^3	p_{3x3}^4
...	...	...	...	...	...	...
		1	$m_{1x}=\Sigma_i\ p_i x_i$	$m_{2x}=\Sigma_i\ p_i x_i^2$	$m_{3x}=\Sigma_i\ p_i x_i^3$	$m_{4x}=\Sigma_i\ p_i x_i^4$

$$= k_{1X}t + k_{2X}t^2/2! + k_{3X}t^3/3! + k_{4X}t^4/4![+t] \ldots$$

where k_{rX} represent the r^{th} cumulant of X. The relationship between the first four cumulants and moments is given by

$$k_{1X} = m_{1X}$$

$$k_{2X} = m_{2X} - m_{1X}^2$$

$$k_{3X} = m_{3X} - 3m_{2X}m_{1X} + 2m_{1X}^3$$

$$k_{4X} = m_{4X} - 4m_{3X}m_{1X} - 3m_{2X}^2 + 12m_{2X}m_{1X}^2 - 6m_{1X}^4$$

These cumulants can be used to calculate the following familiar distributional characteristics (parameters) for x:

Mean $\mu_X = k_{1X}$

Standard Deviation $\sigma_X = \text{sqrt}(k_{2X})$

Coefficient of Skewness $\gamma_X = k_{3X} /(k_{2X})^{3/2}$

Coefficient of Excess Kurtosis $\kappa_X = k_{4X} /(k_{2X})^2$

If N consecutive bets are made, then the total profit, T, is a random variable.

$$T = X_1 + X_2 + \ldots. + X_N$$

where X_i is the outcome on the i^{th} bet.

Assuming that the outcome from each bet is independent of the others, probability theory tells us that the MGF of random variable T is the product of MGFs of the X_i's:

$$M_T(t) = E(\exp(X_1 + X_2 + \ldots. + X_N)t)$$

$$= E(\exp(X_1 t))\ E(\exp(X_2 t)). \ldots E(\exp(X_N t))$$

$$= M_{X1}(t)M_{X2}(t). \ldots M_{XN}(t)$$

If the bets are all on the same game and are all the same size, then the distribution of the profit from each bet is identical, and we obtain an important simplification:

$$M_T(t) = [M_X(t)]^N$$

Taking logarithms, we obtain a relationship between the CGFs:

$$K_T(t) = NK_X(t)$$

This relationship can be expressed in terms of the individual cumulants:

$$k_{rT} = Nk_{rX} \qquad \text{for all } r \geq 1$$

Thus, the cumulants of the total profit after N bets of the same size on a single game can be computed directly from the cumulants of the profit for a single bet.

Likewise, the parameters of T are directly related to the parameters of X:

Mean $\mu_T = N\mu_X$

Standard Deviation $\sigma_T = \sqrt{N}\sigma_X$

Coefficient of Skewness $\gamma_T = \gamma_X/\sqrt{N}$

Coefficient of Excess Kurtosis $\kappa_T = \kappa_X/N$

Distribution of profits

There are several methods to obtain the probabilities of ending up with certain amounts of profit after different numbers of trials, which is also referred to as the distribution of profits. Simulation methods could be used to obtain approximation results, with the accuracy depending on the number of simulation runs. Algebraic computation can produce exact results, but requires computation time when the number of trials is "large." When the number of outcomes in a single bet is two (Win or Lose), the binomial formula can be used to calculate the exact distribution of profits after N bets.[8]

Approximate algebraic methods can be applied effectively, such as the normal approximation to the binomial distribution, which utilizes the mean and the standard deviation.[9] A better normal approximation method is the Normal Power Approximation, which utilizes the coefficients of skewness and excess kurtosis, as well as the mean and standard deviation. This is detailed below:

Let Z be a standardized variable, such that $Z = (T - \mu_T)/ \sigma_T$.

Variable Z has mean 0 and variance 1. Due to the symmetry of the normal distribution, variable Z has a skewness and excess kurtosis of 0. The Normal Power approximation for the cumulative distribution function (CDF) of Z, as given in Pesonen,[10] can be expressed in the following form:

$$\text{Prob } (Z \geq z) = F(z) \text{ approx } \Phi(y)$$

where $\Phi(.)$ is the cumulative normal distribution and

$$y = z - 1/6\gamma_T(z^2 - I) + [1/36\gamma_T{}^2(4z^3 - 7_z) - 1/24\kappa_T\,(z^3 - 3z)]$$

Using y instead of z in the cumulative normal distribution provides improved accuracy when the distribution has skewness and excess kurtosis different from that of a standard normal distribution.

---

8 PACKEL, *supra* note 3.
9 *Id.*
10 Erkki Pesonen, *NP-technique as a Tool in Decision Making*, 8 ASTIN BULLETIN 359–363 (1975), *available at* <http://www.casact.org/library/astin/vol8no3/359.pdf>.