THE MATHEMATICS OF CONFLICTS

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About the Author

Tristan has three degrees; Bachelor of Science (BSc) majoring in mathematics from Macquarie University, Graduate Diploma in Operations Research (GradDipOR) from the University of Technology Sydney, and a PhD in applied mathematics from Swinburne University. Tristan was first introduced to the mathematics of conflicts in undertaking the BSc in a course Operations Research II - STAT379 where game theory was introduced. This led to undertaking the GradDipOR as this was the only university in Sydney that offered a Graduate Diploma in Operations Research. However, there was no game theory taught in this degree. The University of Technology Sydney have two subjects on game theory within microeconomics. The author researched all subjects amongst Australian universities that offer game theory, and it is rarely taught within mathematics or operations research; but rather within microeconomics. Tristan's PhD in applied mathematics was primarily on tennis modelling and game theory was included in optimizing tennis resources. Essentially Tristan self-taught himself game theory by purchasing a book 'Game Theory and Strategy' by Philip Straffin. Tristan has published four papers using game theory but more significantly published a paper 'Applying the Kelly Criterion to Lawsuits' where essentially lawsuits can be modelled like a casino game. This paper is not on game theory but very much related to the mathematics of conflicts, and hence this information was then applied to game theory when there is risk involved. This information is outlined throughout this book and a Kelly Equilibrium is introduced as an alternative to the Nash Equilibrium when there is risk involved in two-person zero-sum games.

Preface

In March/April 2013 the author had a mystical encounter involving a bird as an out-of-body experience after making the connection that 'Yoga is the most effective method to resolving conflicts'. This is documented in 'The Book of Tristan' (pdf) which outlines the authors life story, an ideology with associated policies based on yogic principles, a lifestyle to obtain constant happiness, the truth about Jesus and a solution to obtaining world peace. Although yoga may be the most effective method to resolving conflicts; this book is about the mathematics of conflicts and is generally associated with game theory. However, the author has shown that the mathematics of conflicts is not only applied to game theory analysis; as was the case when the author was involved in an actual legal dispute and applied the well-known Kelly Criterion when multiple outcomes exist (as typically used in casino games) to lawsuits to determine whether it is beneficial to file a lawsuit given there are risks involved if unsuccessful in court. The Kelly Criterion can also be used to determine whether it is worth having legal representation given the additional costs (although the chances of winning the lawsuit may be greater), a suitable amount for an out-of-court settlement through negotiation and a fair arbitration value. This is covered in chapter 2 of this book. Chapter 1 covers information on risk theory by providing the analysis of casino games, the Kelly Criterion when multiples outcomes exists and an application to video poker. Chapter 3 covers warfare conflicts by applying the well-defined hierarchical scoring structure in tennis to the hierarchical structure in warfare to optimize resources. Chapter 4 covers the underlying theory for two-person zero-sum games and provides applications to hierarchical games, tennis and poker. Chapter 5 covers information for two-person zero-sum games when there is risk involved. Risk-averse strategies are devised as well as the Kelly Equilibrium, which can be used as an alternative to the Nash Equilibrium.

Chapter 1
Risk theory

1.1 Introduction

Is it generally accepted that the mean and standard deviation of a random variable give sufficient characteristics for a particular distribution. Although the standard deviation is a measure of risk, it is equally important to obtain two other measures of risk namely the coefficients of skewness and kurtosis (particularly when the distribution is not a Normal Distribution). These four distributional characteristics can be applied directly to the Normal Power approximation to give the distribution of a random variable. We choose to analyze casino games since this particular form applies to lawsuits as given in chapter 2 and two-person zero-sum games when there is risk involved as given in chapter 5.

1.2 Casino games

A casino game can be defined as follows: There is an initial cost C to play the game. Each trial results in an outcome O_i, where each outcome occurs with profit x_i and probability p_i. A profit of zero means the money paid to the player for a particular outcome equals the initial cost. Profits above zero represent a gain for the player; negative profits represent a loss. The probabilities are all non-negative and sum to one over all the possible outcomes. Given this information, the total expected profit $\sum E_i = \sum p_i x_i$. The *percent house margin* (%HM) is then $-\sum E_i/C$. Positive percent house margins indicate that the gambling site on average makes money and the players lose money. Table 1.1 summarizes this information when there are K possible outcomes.

Outcome	Profit	Probability	Expected Profit
O_1	x_1	p_1	$E_1=p_1x_1$
O_2	x_2	p_2	$E_2=p_2x_2$
O_3	x_3	p_3	$E_3=p_3x_3$
…	…	…	…
O_K	x_K	p_K	$E_K=p_Kx_K$
		1.0	**$\sum E_i$**

Table 1.1: Representation in terms of expected profit of a casino game with K possible outcomes.

The outcome or profit from a single bet, X, is a random variable. From probability theory, the *moment generating function* (MGF) of X is

$$M_X(t) = E(\exp(Xt))$$
$$= 1 + m_{1X}t + m_{2X}t^2/2! + m_{3X}t^3/3! + m_{4X}t^4/4! + \ldots,$$

where m_{rX} represent the r^{th} moment of X. The moments of X are readily calculated using
$m_{1X} = \Sigma_i\ p_i\ x_i$
$m_{2X} = \Sigma_i\ p_i\ x_i^2$
$m_{3X} = \Sigma_i\ p_i\ x_i^3$
$m_{4X} = \Sigma_i\ p_i\ x_i^4$
and so on. The calculation of these moments is illustrated in Table 1.2.

The cumulant generating function (CGF) of X is the natural logarithm of the MGF:

$$K_X(t) = \log_e (M_X(t)) = k_{1X}\, t + k_{2X}\, t^2/2! + k_{3X}\, t^3/3! + k_{4X}\, t^4/4! + \ldots,$$

where k_{rX} represent the r^{th} cumulant of X. The relationship between the first four cumulants and moments is given by
$k_{1X} = m_{1X}$
$k_{2X} = m_{2X} - m_{1X}^2$
$k_{3X} = m_{3X} - 3m_{2X}m_{1X} + 2m_{1X}^3$ and
$k_{4X} = m_{4X} - 4m_{3X}m_{1X} - 3m_{2X}^2 + 12m_{2X}m_{1X}^2 - 6m_{1X}^4$.

These cumulants can be used to calculate the following familiar distributional characteristics (parameters) for X:

Mean	$\mu_X = k_{1X}$
Standard Deviation	σ_X = square root of k_{2X}
Coefficient of Skewness	$\gamma_X = k_{3X} / (k_{2X})^{3/2}$
Coefficient of Excess Kurtosis	$\kappa_X = k_{4X} / (k_{2X})^2$.

Outcome	Profit	Probability	1st Moment	2nd Moment	3rd Moment	4th Moment
O_1	x_1	p_1	p_1x_1	$p_1x_1^2$	$p_1x_1^3$	$p_1x_1^4$
O_2	x_2	p_2	p_2x_2	$p_2x_2^2$	$p_2x_2^3$	$p_2x_2^4$
O_3	x_3	p_3	p_3x_3	$p_3x_3^2$	$p_3x_3^3$	$p_3x_3^4$
...	...	...	...	...	...	...
O_K	x_K	p_K	p_Kx_K	$p_Kx_K^2$	$p_Kx_K^3$	$p_Kx_K^4$
		1.0	$\mathbf{m_{1X}= \Sigma_i\ p_ix_i}$	$\mathbf{m_{2X}= \Sigma_i\ p_ix_i^2}$	$\mathbf{m_{3X}= \Sigma_i\ p_ix_i^3}$	$\mathbf{m_{4X}= \Sigma_i\ p_ix_i^4}$

Table 1.2: Representation of the first four moments of the profit of a casino game after one bet.

1.3 Kelly Criterion

The well-established classical Kelly criterion is given by the following result:
Consider a game with two possible outcomes: win or lose, that is played over a "large" number of trials. Suppose the player profits k units for every unit wager and the probabilities of a win and a loss are given by p and q respectively. Further, suppose that on each trial the win probability p is constant with p+q=1. If kp-q>0, so the game is advantageous to the player, then the optimal fraction of the current capital to be wagered to maximize the long-term growth of the bank is given by b*= (kp-q)/k.

Consider the following example: A player profits \$2 with probability 0.35 and profits -\$1 with probability 0.65, as represented by Table 1.3. Since the total expected profit of 2 x 0.35-0.65=0.05>0, the game is advantageous to the player and the optimal fraction is given by b*= (2 x 0.35-0.65)/2 = 0.025. If a player has a \$100 bankroll, then wagering 100*0.025 = \$2.50 on the next hand will maximize the long-term growth of the bank. If the player loses \$1 on that hand, then under the classical Kelly criterion, the next wager should be exactly 99*0.025 = \$2.475. Since fractions are often not allowed in gambling games, this figure should be rounded down to an allowable betting amount.

Outcome	Profit	Probability	Expected Profit
Win	\$2	0.35	\$0.70
Lose	-\$1	0.65	-\$0.65
		1.0	**0.05**

Table 1.3: A sample casino game to determine the optimal betting fraction under the classical Kelly criterion

The Kelly criterion when multiple outcomes (more than two) exist is given by the following Theorem:

Theorem 1.1:
Consider a game with m possible discrete finite mixed outcomes. Let k_i be the profit in the i^{th} case which occurs with probability p_i for $1 \le i \le m$, where at least one outcome is negative and at least one outcome is positive. Without loss of generality let the maximum possible loss (MPL) = $-k_1$. Then if $\sum_{i=1}^{m} k_i p_i > 0$ a winning strategy exists, and the maximum growth of the bank is attained when the proportion of the bank bet at each turn, b, is the smallest positive root of

$\sum_{i=1}^{m} \frac{k_i\, p_i}{-k_1+k_i b} = 0$, in the range ($0 < b < 1$).

Proof of Theorem 1.1:
Assume a constant proportion b of the bank is bet, with m discrete finite mixed outcomes. Let $B(1) / B(0)$ equal $1+ k_i b$ with probability p_i for i = 1 to m, where B(t) represents the player's bank at time t. Assume the player wishes to maximize $g(b) = E[\log(B(1) / B(0))] = \sum_{i=1}^{m} pi \log(1 + kib)$. Without loss of generality let k_1 be the maximum possible loss. In the interval $0< b <-1/k_1$, $1+k_i b > 0$ since $k_i \ge k_1$ for i = 1 to m, so the logarithm of each term is real. Taking derivatives with respect to b,

$$\frac{dg(b)}{db} = \sum_{i=1}^{m} \frac{k_i p_i}{1+k_i b} = g'(b)$$

and

$$\frac{d^2 g(b)}{db^2} = -\sum_{i=1}^{m} \frac{k_i^2 p_i}{(1+k_i b)^2} = g''(b)$$

Note that
(a) $g(0) = 0$,
(b) $g'(0) > 0$ follows directly from the requirement for a winning strategy (so you should bet something), and
(c) $g''(b) < 0$ for $0<b<-1/k_1$ (where k_1 is the MPL) so the first derivate has at most one zero root in this interval.

Hence whenever there is a winning strategy, the force of growth has a unique maximum given by the root of

$$\sum_{i=1}^{m} \frac{k_i\ p_i}{-k_1 + k_i b} = 0$$

Let g(b) represent the rate of growth of the bank which is the quantity to be maximized. Figure 1.1 shows a graphical representation of the Kelly Criterion for the classical case (left) and when multiple outcomes exist (right). The player's bank will grow as long as g(b) > 0, and is maximized when g'(b)=0 (which is represented by g(b*) in Figure 1.1). It is important to note that a player's bank will not grow (and likely to hit ruin) when over betting the bankroll, even though the game is still favourable. This is represented on the graph for the values of b such that g(b) < 0.

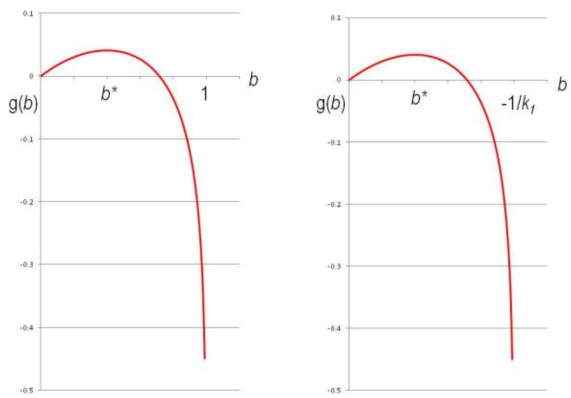

Figure 1.1: Graphical representation of the Kelly criterion for the classical case (left) and when multiple outcomes exist (right), where the optimal betting fraction of b* occurs at a maximum turning point on g(b). Value k_1 is the maximum possible loss in the multiple outcome game.

1.4 Video poker

Video Poker is based on the traditional card game of Draw Poker. Each play of the Video Poker machine results in 5 cards being displayed on the screen from the number of cards in the pack used for that particular type of game (usually a standard 52 card pack or 53 if the Joker is included as a wild card). The player decides which of these cards to hold by pressing the hold button beneath the corresponding cards. The cards that are not held are randomly replaced by cards remaining in the pack. The final 5 cards are paid according to the payout table for that particular type of game. The pay tables follow the same order as traditional Draw Poker. For example, a Full House pays more than a Flush. Without a thorough understanding of video poker, it should be clear in the analysis to follow on how the Kelly Criterion with multiple outcomes can be applied to determining an optimal bet size.

A pay table for the outcomes, profits, probabilities and expected profits for a Jacks or Better machine (known as "All American Poker") are given in Table 1.4. The probabilities were obtained using WinPoker (a commercial product available from the web www.zamzone.com) and assume the player is always maximizing the expected profit on determining the correct playing strategies. Note that $1 is bet each game. It shows that the overall payback for this machine by playing an optimal strategy is 100.72%. The standard deviation is calculated as $5.18. The Kelly Criterion is applied to determine a bet size for this Video Poker game, by using the profits and probabilities given in Table 1.4 and represented in column 5. The solver function in Excel is used to calculate this value as b*=0.030679%. Example: With a $10,000 bankroll, the Kelly Criterion suggests that the player should initially bet $3.07 (likely to be round down to $3).

Outcome	Profit ($)	Probability	Expected Profit ($)	Kelly
Royal Flush	799	1 in 43,450	0.018	0.0148
Straight Flush	199	1 in 7,053	0.028	0.0266
Four of a Kind	39	0.00225	0.088	0.0867
Full House	7	0.01098	0.077	0.0767
Flush	7	0.01572	0.110	0.1098
Straight	7	0.01842	0.129	0.1287
Three of Kind	2	0.06883	0.138	0.1376
Two Pair	0	0.11960	0.000	0.0000
Jacks or Better	0	0.18326	0.000	0.0000
Nothing	-1	0.58076	-0.581	-0.5809
		1	**0.0072**	**0**

Table 1.4: The profits and probabilities for the "All American Poker" game

Chapter 2
Lawsuits

2.1 Introduction

Our daily lives consist of a variety of risk-taking games. They may involve the risk-taking games of blackjack, video poker, sports betting, horse racing or the stock market. There are the less recognized but important risk-taking games of insurance, lawsuits and business. Hence, we are often faced with a set of risk-taking games and a reasonable objective across all these games is to increase our current wealth i.e. grow the size of the bank.

Mathematics is fundamental to solving many problems in industry. Due to the complexity involved in solving industry problems, it can be insightful to break down the problems by finding analogs in games where the mathematics is well-defined. Casino mathematics is one such analog. For example, Barnett and Clarke (2004) found applications to quiz shows by analyzing progressive jackpots in video poker machines. Casino mathematics is an example of decision-making under risk where the probabilities can be obtained exactly and the distribution of payouts after a number of trials can be accurately obtained. Therefore, the player is completely aware of the risks involved with the outcomes of the game. When the game is favourable to the player, the famous Kelly Criterion formula (Kelly, 1956) can be used to maximize the long-term growth of the bank. In some gambling games, such as sports betting and horse-racing, the probabilities are based on estimates, and therefore caution should be taken when applying the Kelly Criterion to a favourable game. Despite this caution, the Kelly Criterion has been applied successfully to blackjack since Edward Thorp's revolutionary blackjack system (Thorp, 1966), and later to success on the stock market (Thorp and Kassouf, 1967). A detailed account of the story and real-world success behind the Kelly Criterion can be found in Poundstone (2005).

There are many decisions involving risk and uncertainty in industry. One example is whether it is worthwhile to file a lawsuit given there are risks involved from legal fees if unsuccessful in the court trial, how much to negotiate if an out-of-court settlement is a possibility and what is a "fair" arbitration value. To obtain insight to the decision-making process to this problem, a model is developed that is representative of the structure used in casino games, which utilizes the Kelly Criterion.

2.2 Dispute resolution

Litigation is a lawsuit filed in a court seeking a legal remedy to the question or dispute existing between the plaintiff and the defendants. The defendants are required to respond to the complaint of the plaintiff. If the plaintiff is successful, judgment will be given in the plaintiff's favour, often resulting in a monetary payout. To avoid the litigation process and hence reduce legal costs between both parties in dispute, a negotiation process may take place to attempt an out-of-court settlement. The other processes in dispute resolution are mediation and arbitration. In mediation, a third party neutral, known as the mediator, assists the parties in formulating their own resolution of the dispute. Arbitration is an adversarial process whereby an independent third party, after hearing submissions from the disputants makes an award binding upon the parties.

Disputes can arise from work agreements and requires the victim (employee or contractor in a work context) to recover money from the injurer (business or company). The example used in this article is based on an actual work agreement that was compiled by the organization and signed by both parties. The following was documented in the work agreement:

Employer: (name of the company)
Employee: (name of the employee)
Terms & Conditions of Employment:
1. Commencement Date:
The date for commencement of duties is Monday 17th July 2006
2. Remuneration
a) Position Hours
The position will be based on 0.6 of an equivalent full time position
b) Remuneration
On appointment your remuneration will be $500.00 per week which will be paid fortnightly
3. Probationary Period
A probationary period of three months will apply.

The victim was under the impression that he/she was an employee of the company and hence superannuation and holiday pay would apply. When issues were brought up about the type of agreement in 2008, the company stated that he/she was an independent contractor. The company had not issued any tax forms and no tax was taken out. The wages were invariably late, forcing the victim to eventually hand in his/her resignation and be out of the work force. The amount for each query is given in Table 2.1 and shows the total disputed amount of $13,000. The situation is complex and many people in the work force would not know the processes involved to best recover the money. Litigation and negotiation processes are now addressed.

Outcome	Query	Amount ($)
1	Holiday Pay	1,000
2	Late Payments	1,000
3	Superannuation	5,000
4	Out of Work	6,000
	Total	**13,000**

Table 2.1: Type of query with the associated amount for an employment dispute

2.3 Litigation

The victim is considering filing a lawsuit against the injurer in an attempt to obtain the total disputed amount of $13,000. There are risks involved in going to court if unsuccessful. The victim's chances of recovering the money would likely increase with legal representation. However, there are additional legal costs associated with this likely increase in success. It is therefore important to analyze both situations where the victim is representing themselves in court and when a lawyer is acting on the victim's behalf. The total legal cost with legal representation is estimated to be $1,800 and by the victim representing themselves in court, the legal cost is $300. Table 2.2 represents the situation where the victim is represented by a lawyer in court and the structure is in the form of a casino game, as outlined in Chapter 16. The profits are obtained from the amounts given in Table 2.1. For example, Outcome 1 was obtained by the amount obtained for Holiday Pay less the legal costs ($1,000-$1,800=-$800) and Outcome 1,2 was obtained by adding the amounts for Holiday and Late Payments less the legal costs ($1,000+$1,000-$1,800=$200). The associated probabilities for each outcome are estimated and in reality, could be based on historical data. There are obvious constraints on the probabilities such that the probability for Outcome 1,2,3,4 is less than the probability for Outcome 1,2,3 which is less than the probability of Outcome 1,2 which is less than the probability for Outcome 1. Other constraints on probabilities also apply. All probabilities must be positive and the sum of the probabilities must equal 1. The game is favourable to the victim with a total expected profit of $2,530 and could therefore consider filing a lawsuit. However, there are risks involved given a 44% chance of ending up with a loss and a 32% chance of losing $1,800. A procedure using the Kelly criterion is now given to assist the victim with the decision as to whether to file a lawsuit.

The Kelly Criterion in a gambling context assumes that a player bets a proportion of their current bankroll and over betting can potentially lead to ruin. Suppose the amount that a player is allowed to bet on each trial is fixed according to the playing rules of the game. The Kelly Criterion can still be applied by determining the minimum bankroll requirements such that the player is not over betting. If A represents the fixed amount to bet on each trial and B represents the player's current bankroll, then a player would not be over betting in the game only if

$$B \geq A / b^* \qquad (1)$$

For example, in the "All American Poker" game from Table 1.4 was fixed at a betting amount of $2.50 for each trial, then a player would not be over betting in the game if B ≥ 2.5 / 0.030679% = $8,148.90.

In the context of litigation (as in the game given by Table 2.2) the amount that the victim is allowed to bet is fixed by the total legal costs and would remain fixed even if the game was played over many trials. The total legal costs are given by the maximum possible loss (MPL) in the representation of the game. Given the well-defined mathematics of the Kelly criterion, the victim's decision as to whether to file a lawsuit could be based on Equation (1). Using Solver in Excel, b*=0.460. Therefore, the victim may consider filing a lawsuit against the injurer if their bankroll is greater than or equal to 1800/0.460 = $3,915. In general, the victim may consider filing a lawsuit against the injurer if their bankroll is greater than or equal to MPL/b*. Note that a bankroll is gambling or risk money (that you can afford to lose), as opposed to the money you live on.

Given that the Kelly Criterion method is intended for a "large" number of trials, it should not be too much of a concern that lawsuits usually require only 1 or 2 trials. Firstly, the victim (player) may be involved in many favourable risk-taking games such as the stock market, horse racing, blackjack and litigation, which collectively involve many trials. The Kelly Criterion can still apply in the litigation context on just the one trial. From an investment perspective, it is important not to hit ruin or be close to losing your bankroll if the worst case happens i.e. lose the court trial. Therefore, the victim's bankroll needs to be greater than the maximum possible loss. How much greater? That of course can depend on many factors but as a guide or to provide an objective formula, the optimal fraction given by the Kelly Criterion is a reasonable estimate since this optimal fraction will be "considerably" less than one and takes into account the full distribution of the game. Also note that the Kelly Criterion is most sensitive to the MPL (the total legal costs) and therefore the probabilities of obtaining the various outcomes for the various queries do not need to be 100% accurate.

Outcome	Profit ($)	Probability	Expected Profit ($)
1	-800	0.06	-48
1,2	200	0.05	10
1,2,3	5,200	0.04	208
1,2,3,4	11,200	0.03	336
1,2,4	6,200	0.04	248
1,3	4,200	0.05	210
1,3,4	10,200	0.04	408
2	-800	0.06	-48
2,3	4,200	0.05	210
2,3,4	10,200	0.04	408
2,4	5,200	0.05	260
3	3,200	0.06	192
3,4	9,200	0.05	460
4	4,200	0.06	252
Lose	-1,800	0.32	-576
		1	**2,530**

Table 2.2 The outcomes of a lawsuit game with legal representation in court

Table 2.3 represents the situation where the victim is representing themselves in court. The game is favourable to the victim with a total expected profit of $2,920, which is greater than the total expected profit of $2,530 given in Table 2.2. However, the chances of winning the lawsuit are less than the game given by Table 2.2, where the victim has a 52% chance of winning a positive amount without legal representation compared to a 56% chance of winning a positive amount with legal representation. Using Solver in Excel, b*=0.472. Therefore, the victim may consider filing a lawsuit against the injurer if their bankroll is greater than or equal to 300/0.472 = $635. It is important to understand the differences in the games given by Tables 2.2 and 2.3, and often situations arise where obtaining higher expected costs have more associated risks involved.

Outcome	Profit ($)	Probability	Expected Profit ($)
1	$700	0.05	$35
1,2	$1,700	0.04	$68
1,2,3	$6,700	0.03	$201
1,2,3,4	$12,700	0.02	$254
1,2,4	$7,700	0.03	$231
1,3	$5,700	0.04	$228
1,3,4	$11,700	0.03	$351
2	$700	0.05	$35
2,3	$5,700	0.04	$228
2,3,4	$11,700	0.03	$351
2,4	$6,700	0.02	$134
3	$4,700	0.05	$235
3,4	$10,700	0.04	$428
4	$5,700	0.05	$285
Lose	-$300	0.48	-$144
		1	**$2,920**

Table 2.3: The outcomes of a lawsuit game without legal representation in court

2.4 Negotiation and Arbitration

After the victim has sent a Letter of Demand or taken legal action by filing a lawsuit with the Magistrates Court, the injurer may want to negotiate an out-of-court settlement. According to the Von Neumann-Morgenstern concept of a utility function (Winston, 1994), there is a lottery such that the victim would be indifferent between a payout of $x and the game given by the payouts with the associated probabilities as represented in Tables 2.2 and 2.3. This value of $x could be interpreted in the legal field as an out-of-court settlement by negotiation.

We will assume that the victim has an adequate bankroll for the game given in Table 2.2 and has taken legal action by sending a Letter of Demand. The aim is to show that the victim should be willing to accept an amount by negotiation which is less than the total expected profit of $2,530. The Kelly Criterion is sensitive to the MPL and the objective is to maximize the long-term growth of the bank. Minimizing the probability of the MPL can increase the long-term growth of the bank even though the expected profit may be reduced. Maximizing the long-term growth of the bank is equivalent to maximizing the total expected bank. Table 2.4 gives the expected bank based on the profit outcomes with the associated probabilities from Table 2.2, with b* = 0.460. For example, the expected bank for Outcome 1 was obtained by -$800 x 0.06 x 0.460 = -$22. Therefore, the total expected bank = total expected profit x b*. The total expected bank could be used as the minimum amount that the victim should be willing to accept for an out-of-court settlement, and given as $1,163 from Table 2.4. Also, more importantly the total expected bank can also be used as an arbitration amount for resolving the dispute.

Outcome	Profit ($)	Probability	Expected Bank ($)
1	-800	0.06	-22
1,2	200	0.05	5
1,2,3	5,200	0.04	96
1,2,3,4	11,200	0.03	154
1,2,4	6,200	0.04	114
1,3	4,200	0.05	97
1,3,4	10,200	0.04	188
2	-800	0.06	-22
2,3	4,200	0.05	97
2,3,4	10,200	0.04	188
2,4	5,200	0.05	120
3	3,200	0.06	88
3,4	9,200	0.05	212
4	4,200	0.06	116
Lose	-1,800	0.32	-265
		1	**1,163**

Table 2.4: The expected bank of a lawsuit game with legal representation in court

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Chapter 3
Warfare

3.1 Introduction

The Defence Science of Technology Organization (DSTO) chose to analyze tennis as an analog to warfare at the 2003 Mathematics in Industry Study Group, with the aim of using results obtained within tennis, to gain insights that could be used to solve problems related to warfare
(www.unisa.edu.au/misg/Equation free booklet 2003.pdf). By making the following transitions, the models for a tennis match can be transformed into models
for warfare:
skirmish, battle, campaign, war → point, game, set, match
attack/defence→ serve/return
combatants → players
fought → played

3.2 Tennis model

We explain the method by first looking at a single game where we have two players, A and B, and player A has a constant probability p_A of winning a point on serve. We set up a Markov chain model of a game where the state of the game is the current game score in points (thus 40-30 is 3-2). With probability p_A the state changes from a, b to a + 1, b and with probability $q_A=1-p_A$ it changes from a, b to a, b + 1. Thus if $P_A^{pg}(a,b)$ is the probability that player A wins the game when the score is (a,b), we have:

$P_A^{pg}(a,b)=p_AP_A(a+1,b)+q_AP_A(a,b+1)$
The boundary values are:
$P_A^{pg}(a,b) = 1$ if $a = 4, b \le 2$, $P_A^{pg}(a,b) = 0$ if $b = 4, a \le 2$.

The boundary values and formula can be entered on a simple spreadsheet. The problem of deuce can be handled in two ways. Since deuce is logically equivalent to 30-30, a formula for this can be entered in the deuce cell. This creates a circular cell reference, but the iterative function of Excel can be turned on, and Excel will iterate to a solution. In preference, an explicit formula is obtained by recognizing that the chance of winning from deuce is in the form of a geometric series

$P_A^{pg}(3,3) = p_A^2 + p_A^2 2p_Aq_A + p_A^2 (2p_Aq_A)^2 + p_A^2 (2p_Aq_A)^3 + \ldots\ldots\ldots$
where the first term is p_A^2 and the common ratio is $2p_Aq_A$

The sum is given by $p_A^2/(1-2p_Aq_A)$ provided that $-1<2p_Aq_A<1$. We know that $0<2p_Aq_A<1$, since $p_A>0$, $q_A>0$ and $1-2p_Aq_A=p_A^2+q_A^2>0$.

Therefore, the probability of winning from deuce is $p_A^2/(1-2p_Aq_A)$. Since $p_A+q_A=1$, this can be expressed as:

$$P_A^{pg}(3,3) = p_A^2 /(p_A^2+q_A^2)$$

Excel spreadsheet code to obtain the conditional probabilities of player A winning a game on serve is as follows:
Enter **p_A** in cell D1
Enter **q_A** in cell D2
Enter **0.60** in cell E1
Enter **=1-E1** in cell E2
Enter **1** in cells C11, D11 and E11
Enter **0** in cells G7, G8 and G9
Enter **= E1^2/(E1^2+E2^2)** in cell F10
Enter **=$E$1*C8+$E$2*D7** in cell C7
Copy and Paste cell **C7** in cells D7, E7, F7, C8, D8, E8, F8, C9, D9, E9, F9, C10, D10 and E10

Notice the absolute and relative referencing used in the formula =$E$1*C8+$E$2*D7. By setting up an equation in this recursive format, the remaining conditional probabilities can easily and quickly be obtained by copying and pasting.

Table 3.1 represents the conditional probabilities of player A winning the game from various score lines for $p_A = 0.60$. It indicates that a player with a 60% chance of winning a point has a 74% chance of winning the game. Note that since advantage server is logically equivalent to 40-30, and advantage receiver is logically equivalent to 30-40, the required statistics can be found from these cells. Also worth noting is that the chances of winning from deuce and 30-30 are the same.

				B score		
		0	15	30	40	game
	0	0.74	0.58	0.37	0.15	0
	15	0.84	0.71	0.52	0.25	0
A score	30	0.93	0.85	0.69	0.42	0
	40	0.98	0.95	0.88	0.69	
	game	1	1	1		

Table 3.1: The conditional probabilities of A winning the game from various score lines

Similar equations can be developed for when player B is serving such that p_A and p_B represent constant probabilities of player A and player B winning a point on their respective serves. Also $P_A^{pg}(a,b)$ and $P_B^{pg}(a,b)$ represent the conditional probabilities of player A winning a game from point score (a,b) for player A and B serving in the game respectively.

A tennis match consists of four levels - (points, games, sets, match). Games can be standard games (as above) or tiebreak games, sets can be advantage or tiebreak, and matches can be the best-of-5 sets or the best-of-3 sets. To win a set a player needs six games with at least a two game lead. If the score reaches 6 games-all, then a tiebreak game is played in a tiebreak set to determine the winner of the set, otherwise standard games continue indefinitely until a player is two games ahead and wins the set. This latter scoring structure is known as an advantage set and is used as the deciding set in the Australian Open, French Open and Wimbledon. In some circumstances we may be referring to points in a standard or tiebreak game and other circumstances points in a tiebreak or advantage set. It becomes necessary to represent

points in a game as pg,
points in a tiebreak game as pg^T,
points in an advantage set as ps,
points in a tiebreak set as ps^T
points in a best-of-5 set match (advantage fifth set) as pm,
points in a best-of-5 set match (all tiebreak sets) as pm^T
games in an advantage set as gs,
games in a tiebreak set as gs^T,
sets in a best-of-5 set match (advantage fifth set) as sm, and
sets in a best-of-5 set match (all tiebreak sets) as sm^T.

Since the chance of a player winning a tiebreak game depends on who is serving, two interconnected sheets are required, one for when player A is serving and one for when player B is serving. The equations that follow for modelling a tiebreak game, set and match are those for player A serving in the game. Similar formulas can be produced for player B serving in the game.

Let $P_A^{pg^T}(a,b)$ and $P_B^{pg^T}(a,b)$ represent the conditional probabilities of player A winning a tiebreak game from point score (a,b) for player A and player B serving in the game respectively.

Recurrence formulas:
$P_A^{pg^T}(a,b) = p_A P_B^{pg^T}(a+1,b)+q_A P_B^{pg^T}(a,b+1)$, if (a+b) is even
$p_A^{pg^T}(a,b) = p_A P_A^{pg^T}(a+1,b)+q_A P_A^{pg^T}(a,b+1)$, if (a+b) is odd

Boundary values:
$P_A^{pg^T}(a,b)=1$ if $a=7$, $0 \leq b \leq 5$
$P_A^{pg^T}(a,b)=0$ if $b=7$, $0 \leq a \leq 5$
$P_A^{pg^T}(6,6)=p_A q_B/(p_A q_B+q_A p_B)$

where: $q_B=1-p_B$

Table 3.2 represents the conditional probabilities of player A winning the tiebreak game from various score lines for $p_A = 0.62$ and $p_B = 0.60$, and player A serving. Table 3.3 is represented similarly with player B serving.

Note how the calculations are obtained by the interconnection of both sheets. For example

$P_A^{pgT}(0,0) = p_A P_B^{pgT}(1,0)+q_A P_B^{pgT}(0,1)$
$= 0.62 * 0.62 + 0.38 * 0.39$
$= 0.53$

					B score				
		0	1	2	3	4	5	6	7
	0	0.53	0.44	0.29	0.20	0.10	0.04	0.01	0
	1	0.67	0.53	0.43	0.27	0.17	0.07	0.02	0
	2	0.76	0.68	0.53	0.42	0.24	0.13	0.03	0
A score	3	0.87	0.77	0.69	0.53	0.40	0.20	0.08	0
	4	0.93	0.89	0.80	0.72	0.52	0.37	0.13	0
	5	0.98	0.95	0.92	0.83	0.75	0.52	0.32	0
	6	0.99	0.99	0.98	0.96	0.89	0.82	0.52	
	7	1	1	1	1	1	1		

Table 3.2: The conditional probabilities of player A winning the tiebreak game from various score lines for $p_A = 0.62$ and $p_B = 0.60$, and player A serving

					B score				
		0	1	2	3	4	5	6	7
	0	0.53	0.39	0.29	0.17	0.10	0.03	0.01	0
	1	0.62	0.53	0.37	0.27	0.14	0.07	0.01	0
	2	0.76	0.63	0.53	0.35	0.24	0.10	0.03	0
A score	3	0.83	0.77	0.63	0.53	0.33	0.20	0.05	0
	4	0.93	0.86	0.80	0.65	0.52	0.29	0.13	0
	5	0.97	0.95	0.89	0.83	0.67	0.52	0.21	0
	6	0.99	0.99	0.98	0.93	0.89	0.71	0.52	
	7	1	1	1	1	1	1		

Table 3.3: The conditional probabilities of player A winning the tiebreak game from various score lines for $p_A = 0.62$ and $p_B = 0.60$, and player B serving

Formulas are now given for a tiebreak set. Similar formulas can be obtained for an advantage set.

Let $P_A^{gsT}(c,d)$ and $P_B^{gsT}(c,d)$ represent the conditional probabilities of player A winning a tiebreak set from game score (c,d) for player A and player B serving in the game respectively.

Recurrence formula:
$P_A^{gsT}(c,d)=P_A^{pg}(0,0)P_B^{gsT}(c+1,d)+[1- P_A^{pg}(0,0)]P_B^{gsT}(c,d+1)$

Boundary Values:
$P_A^{gsT}(c,d)=1$ if $c=6, 0\le d\le 4$ or $c=7, d=5$
$P_A^{gsT}(c,d)=0$ if $d=6, 0\le c\le 4$ or $c=5, d=7$

$P_A^{gsT}(6,6) = P_A^{pgT}(0,0)$

Notice how the cell $P_A^{pg}(0,0)$, which represents the probability of winning a game, is used in the recurrence formula for a tiebreak set. Using the formulas given for a game and a tiebreak game conditional on the point score and a tiebreak set conditional on the game score, calculations are now obtained for a tiebreak set conditional on both the point and game score as follows.

Let $P_A^{psT}(a,b:c,d)$ represent the probability of player A winning a tiebreak set from (c,d) in games, (a,b) in points and player A serving in the set. This can be calculated by:

$P_A^{psT}(a,b:c,d)=P_A^{pg}(a,b)P_B^{gsT}(c+1,d)+[1-P_A^{pg}(a,b)]P_B^{gsT}(c,d+1)$, if $(c,d) \neq (6,6)$
$P_A^{psT}(a,b:c,d)=P_A^{pgT}(a,b)$, if $(c,d)=(6,6)$

Formulas are now given for a best-of-5 set match, where all sets are tiebreak sets. Similar formulation can be obtained for a best-5 set match, where the deciding fifth set is advantage. Formulation can also be obtained for a best-of-3 set match.

Let $P^{smT}(e,f)$ represent the conditional probabilities of player A winning a best-of-5 set tiebreak match from set score (e,f).

Recurrence Formula:
$P^{smT}(e,f)= P_A^{gsT}(0,0)P^{smT}(e+1,f)+[1-P_A^{gsT}(0,0)]P^{smT}(e,f+1)$

Boundary Values:
$P^{smT}(e,f)=1$ if $e=3, f\leq 2$
$P^{smT}(e,f)=0$ if $f=3, e\leq 2$

Notice how the cell $P_A^{gsT}(0,0)$, which represents the probability of winning a tiebreak set, is used in the recurrence formula for a best-of-5 set match. Using the formulas given for a tiebreak set conditional on the point and game score and a best-of-5 set tiebreak match conditional on the set score, calculations are obtained for a best-of-5 set tiebreak match conditional on the point, game and set score as follows.

Let $P_A^{pmT}(a,b:c,d:e,f)$ represent the probability of player A winning a tiebreak match from (e,f) in sets, (c,d) in games, (a,b) in points and player A serving in the match. This can be calculated by:

$P_A^{pmT}(a,b:c,d:e,f)=P_A^{psT}(a,b:c,d)P^{smT}(e+1,f)+[1-P_A^{psT}(a,b:c,d)]P^{smT}(e,f+1)$

Excel spreadsheet code was given directly to obtain the conditional probabilities of player A winning a game on serve. Similarly, spreadsheet code can be developed for a game with player B serving, tiebreak game, tiebreak set conditional on the game score and a best-of-5 set tiebreak match conditional on the set score. By assigning a value for p_B to a cell (cell E3 for example), the probability of winning a match from the outset can be obtained for any probability value of p_A and p_B by changing the probability values given in cells E1 and E3. By adding additional formulas to the spreadsheet for a tiebreak set conditional on the point and game score and for a best-of-5 set tiebreak match conditional on the point, game and set score; the chances of player's winning the set and match can be obtained conditional on who is currently serving, point score, game score and set score. An interactive tennis calculator to reflect this methodology is available at http://strategicgames.com.au/tennisdeucesim.xlsx.

3.3 Strategy a)

A best-of-3 set match is a contest where the first player to win 2 sets wins the match. Analyzing this system is non-trivial despite its relatively simple structure, because it is not certain that the third set will be played.

Consider the situation where a player has a constant probability p of winning a set. What is the probability of this player winning a best-of- 3 set match? The player can win the match by either winning in straight sets with probability p^2, losing the first set and winning the last two sets with probability $(1-p)p^2$ or winning the first and last set and losing the second set with probability $p(1-p)p$. Summing these, the probability of the player to win the match is given by $p^2(3-2p)$.

Now suppose a player increases his effort for one set at a match score in sets (e,f) (e= referred player's score, f=opponent's score), to change his probability of winning this set from p to $p+\varepsilon$, where $p+\varepsilon < 1$. This is equivalent to the opponent decreasing his effort at a match score (e,f) to change his probability of winning this set from $1-p$ to $1-p-\varepsilon$, since an increase of the probability of winning to one player is a decrease to the other player. On which set, should the player apply the increase to optimize their chances of winning the match? If the increase in effort is applied at (0,0), the probability for the player to win the match becomes $(p+\varepsilon)p(2-p) + (1-p-\varepsilon)p^2 = p^2(3-2p) + \varepsilon 2p(1-p)$. The same result is obtained if an increase in effort is applied at (1,1). Similarly, the probabilities of a player to win the match when an increase in effort is applied at one of (1,0) or (0,1) is $p^2(3-2p) + \varepsilon p(1-p)$. Conditional on the match score reaching (1,0), the probability for a player to win the match when an increase in effort is applied at (1,0) or (1,1) is $p(2-p)+ \varepsilon(1-p)$; and conditional on the match score reaching (0,1), the chance for a player to win the match when an increase in effort is applied at (0,1) or (1,1) is $p^2+ \varepsilon p$. Table 3.4 gives the increase in probability when effort is applied throughout the match. The first set played begins with the match score at (0,0). The third set is played only if the match score reaches (1,1). The second set played occurs with the match score at either (1,0) or (0,1). The probability of a player winning the match when one increase in effort is applied on the first, second or third set played is equal to $p^2(3-2p) + \varepsilon 2p(1-p)$. Using this result and the results represented in Table 3.4, an increase in effort could be applied on any set played within the match, and the player has optimized their chances of winning.

Current match score	Match score at which an increase is applied	Increase in probability of winning match
(0,0)	(0,0) (1,0) (0,1) (1,1)	$\varepsilon 2p(1\text{-}p)$ $\varepsilon p(1\text{-}p)$ $\varepsilon p(1\text{-}p)$ $\varepsilon 2p(1\text{-}p)$
(1,0)	(1,0) (1,1)	$\varepsilon(1\text{-}p)$ $\varepsilon(1\text{-}p)$
(0,1)	(0,1) (1,1)	εp εp
(1,1)	(1,1)	ε

Table 3.4: The increase in probability when effort is applied throughout the match.

3.4 Strategy b)

Now suppose a player adopts a strategy of increasing his effort on the first, second or third set played by ε, and decreases p on the first, second or third set played (but a different set played from that of the increase) by ε, where $0 < p+\varepsilon < 1$. Calculations show the chance of the player winning the match for this situation is equal to p^2 (3-2p)+ ε^2(2p-1). When p=½, 2p-1 = 0, and there is no change in the chances for either player to win the match. When p> ½, the chance for the player to win the match increases by ε^2(2p-1) and therefore the opponent's chances to win the match decrease by ε^2(2p-1). This implies that it is an advantage for the better player to vary his effort whilst maintaining his mean probability of winning a set. It follows by symmetry that the weaker player is disadvantaged by varying his effort.

3.5 Strategy c)

Morris (1977) defines the importance of a point for winning a game (I^{pg}) as the probability that the server wins the game given he wins the next point minus the probability that the server wins the game given he loses the next point. Table 3.5 gives the importance of points to winning the game when the server has a 0.62 probability of winning a point on serve, and shows that 30-40 and Ad-Out are the most important points in the game. In a similar way, we can define the importance of a game to winning a set and the importance of a set to winning a match. Table 3.6 gives the importance of games to winning a tiebreak set (I^{gs}) for player A serving. Player A and Player B were assigned point probabilities of 0.62 and 0.60 respectively to reflect overall averages in men's tennis. It is clear that every point is equally important for both players. Table 3.6 shows that the tiebreak game has the highest importance of 1.00, as the winner of this game wins the set. Similarly, table 3.7 gives the importance of sets to winning a best-of-5 set match (I_{SM}) and shows that the deciding set at 2 sets-all has the highest importance of 1.00, as the winner of this set win the match. Morris (1977) derived the following useful multiplicative result to obtain the importance of a point to winning the match (I^{pm}): For any point of any game of any set, $I^{pm} = I^{pg} * I^{gs} * I^{sm}$.

		Receiver score				
		0	15	30	40	Ad
Server score	0	0.25	0.34	0.38	0.28	
	15	0.19	0.31	0.45	0.45	
	30	0.11	0.23	0.45	0.73	
	40	0.04	0.10	0.27	0.45	0.73
	Ad				0.27	

Table 3.5: Importance of points to winning a game when the server has a 0.62 probability of winning a point on serve

		Player B score						
		0	1	2	3	4	5	6
Player A score	0	0.29	0.29	0.22	0.18	0.06	0.02	
	1	0.26	0.32	0.33	0.21	0.16	0.03	
	2	0.25	0.29	0.36	0.37	0.20	0.11	
	3	0.13	0.27	0.33	0.42	0.43	0.14	
	4	0.08	0.11	0.30	0.38	0.52	0.54	
	5	0.01	0.06	0.08	0.34	0.46	0.52	0.53
	6						0.47	1.00

Table 3.6: Importance of games to winning a tiebreak set when player A and player B have a 0.62 and 0.60 probability of winning a point on service respectively and player A is serving

		B score		
		0	1	2
A score	0	0.36	0.42	0.32
	1	0.32	0.49	0.57
	2	0.18	0.43	1.00

Table 3.7: Importance of sets to winning a best-of-5 set match when player A and player B have a 0.62 and 0.60 probability of winning a point on service respectively

Morris (1977) stated that the importance of any point in a match is equal to the product of the importance of the point in the game, the importance of the game in the set and the importance of the set in the match.

$$I^{pm}(a,b:c,d:e,f) = I^{pg}(a,b)I^{gs}(c,d)I^{sm}(e,f)$$

Morris (1977) also showed as a strategy that a player could increase their chances of winning by increasing effort on the important points and decreasing effort on the unimportant points. He stated, for example, that if a player increased p from 0.60 to 0.61 on the important half of his service points, and decreased from 0.60 to 0.59 on the unimportant half, he would increase his winning percentage for a game by 0.0075 from 0.7357 to 0.7432.

3.6 Strategy d)

We now model the situation in warfare where a a combatant has a “large” number of available increases in effort for use in the war. However, there are costs associated for applying an increase in effort at a particular skirmish (and a reward for winning the war). An increase an effort could be the cost of firing a missile for example. Where should a combatant apply the increases to maximize on the expected payoff throughout the war? For this problem it is assumed there are a large number of increases in effort available for use, and if the allocated M increases run out, the supply can always be replenished. There is a reward R for winning the overall war and a cost C for applying an increase at a particular skirmish. Ultimately the hope is to win the war by applying M increases, to maximize R − MC. There might be a good chance of winning the war by applying an increase on every skirmish, but overall the war might be a financial loss because of the high costs associated with the large number of increases. Clearly if we work in monetary terms there is a trade-off between the value of winning the war and the number of increases in effort that are applied and hence there is risk involved. This trade-off might be less attractive if non-financial considerations are taken into account.

Firstly, consider one level of nesting, such as campaigns within a war, where G = the cost of applying an increased effort to a campaign in the war. Let X be a random variable for the payout at (e,f) in a war with no increase and Y be a random variable for the payout at (e,f) in a war with an increase. Let p represent the probability of combatant A winning a campaign and P(e,f) represent the conditional probabilities of combatant A winning the war from campaign score(e,f).

If E[X] and E[Y] represent the expected payout at (e,f) in a war with no increase and with an increase by ε respectively, then:

$E[X] = [pP(e + 1,f) + (1-p)P(e,f + 1)]R$ $E[Y] = [(p + \varepsilon)P(e + 1,f) + (1-p-\varepsilon)P(e,f + 1)]R-G$

If $E[Y]-E[X] > 0$ an increase should be applied at (a,b).

$E[Y]-E[X]$ simplifies to $R^2[P(e+1,f)-P(e,f+1)]-G$, which is equivalent to $R^2I(e,f)-G$.

This implies that an increase should be applied at (e,f) if $R^2I(e,f)-G > 0$, or equivalently if:

$$I(e,f) > G/R\varepsilon$$

$\varepsilon I(a,b)$ represents the increased chance of winning the war by applying an increase in effort at (a,b). The positive component of the expected payout then becomes $R^2I(e,f)$. However there is a cost G for applying an increase in effort at (e,f). The negative component of the payout then becomes$-G$, and the total payout is $R^2I(e,f)-G$.

Extending this analysis to 3 levels of nesting (skirmishes, battles, campaigns, war), an increase should be applied at (a,b : c,d : e,f) if:

$$I(a,b : c,d : e,f) > C/R\varepsilon$$

Therefore:

$$I(a,b)I(c,d)I(e,f) > C/R\varepsilon$$

A spreadsheet calculator can be obtained for this warfare model from http://strategicgames.com.au/warfare.xlsx

Chapter 4
Two-person zero-sum games

4.1 Introduction

Note the information covered in sections 4.1, 4.2, 4.3, 4.4, 4.5 and 4.8 has been taken from Straffin (1993).

Game theory is the logical analysis of situations of conflict and cooperation. More specifically, a game is defined to be any situation in which
i) There are at least two players. A player may be an individual, but it may also be more general entity like a company, a nation, or even a biological species.
ii) Each player has a number of possible strategies, course of action which he or she may choose to follow.
iii) The strategies chosen by each player determine the outcome of the game.
iv) Associated to each possible outcome of the game is a collection of numerical payoffs, one to each player. These payoffs represent the value of the outcome to the different players.

Consider Game 4.1 which represents a two-person nonzero-sum game
There are two players, P1 and P2
P1 has 3 strategies; A, B and C
P2 has 2 strategies; A and B
If P1 chooses strategy A and P2 chooses strategy B, then P1 will receive a payoff of
-3 and Player 2 will receive a payoff of 3

		P2	
		A	B
	A	(2,-2)	(-3,3)
P1	B	(0,0)	(2,-2)
	C	(-5,5)	(10,-10)

Game 4.1

4.2 Dominance

A two-person zero-sum game where P1 has *m* strategies and P2 has *n* strategies can be represented by an *m* x *n* array of numbers, giving the payoffs from P2 to P1 for each of the *m.n* possible outcomes. Such an array is called an *m* x *n* matrix, so these games are also known as matrix games.

DEFINITION. A strategy S dominates a strategy T if every outcome in S is at least as good as the corresponding outcome in T, and at least one outcome in S is strictly better than the corresponding outcome in T.

DOMINATED PRINCIPLE. A rational player should never play a dominated strategy.

Consider Game 4.2 which represents a two-person zero sum game. Note that it is enough to give the payoff to P1 for each outcome, since the payoff to P2 will just be the corresponding negative. Strategy C dominates strategy B for P2. Therefore, according to the DOMINATED PRINCIPLE Game 4.2 can be reduced to Game 4.3.

			P2	
		A	B	C
	A	2	0	4
P1	B	1	2	3
	C	4	1	2

Game 4.2

		P2	
		A	B
	A	2	0
P1	B	1	2
	C	4	1

Game 4.3

From Game 4.3 strategy A is dominated by strategy C for P1 (note this is not the case in Game 4.2). Therefore, according to the DOMINATED PRINCIPLE Game 4.3 can be reduced to Game 4.4.

		P2	
		A	B
P1	B	1	2
	C	4	1

Game 4.4

4.3 Saddle points

DEFINITION. An outcome in a matrix game (with payoffs to the row player) is called a saddle point if the entry at that outcome is both less than or equal to any entry in its row, and greater than or equal to any entry in its column

SADDLE POINT PRINCIPLE. If a matrix game has a saddle point, both players should play a strategy which contains it.

DEFINITION. For any matrix game, if there is a number *v* such that P1 has a strategy which guarantees that they will win at least *v*, and P2 has a strategy which guarantees that P1 will win no more than *v*, then *v* is called the value of the game.

We need a way of telling whether a game has a saddle point, and find the saddle point (or points) if it does. The method, illustrated below in Game 4.5, is first to write down the minimum entry in each row, and strikethrough the maximum of these row minima. Then write down the maximum entry in each column, and circle the minimum of these column maxima. If the maximin of the rows and the minimax of the columns are the same, then they appear at saddle point strategies. In this example, the two saddle points are P1: A or C, P2: C. Game theorists prescribe

- 2 as the value of the game
- A or C as P1's optimal strategy
- C as P2's optimal strategy

			P2				
		A	B	C	D	Row minimum	
	A	4	3	~~2~~	5	2	← maximin
P1	B	-10	2	0	-1	-10	
	C	7	5	~~2~~	3	2	← maximin
	D	0	8	-4	-5	-5	
Column maximum		7	8	2	5		
				↑ minimax			

Game 4.5

4.4 Mixed strategies

Consider Game 4.6. Since there is no saddle point in this game, neither player would want to play a single strategy with certainty, for the other player could take advantage of such a choice. The only sensible plan is to use some random device to decide which strategy to play. Such a plan, which involves playing a mixture of strategies according to certain fixed probabilities, is called a mixed strategy. The contrasting plan of playing one strategy with certainty is called a pure strategy.

		P2			
		A	B	Row minimum	
P1	A	2	-3	-3	
	B	0	3	0	← maximin
Column maximum		2	3		
		↑ minimax			

Game 4.6

DEFINITION. The expected value of getting payoffs $a_1, a_2, \ldots\ldots a_k$ with respective probabilities $p_1, p_2, \ldots\ldots, p_k$ is $p_1a_1 + p_2a_2 + \ldots\ldots + p_ka_k$.

EXPECTED VALUE PRINCIPLE. If you know that your opponent is playing a given mixed strategy, and will continue to play it regardless of what you do, you should play your strategy which has the largest expected value.

Suppose that P2 plays a mixed strategy with probabilities p for A, (1-p) for B, where p is some number between 0 and 1. P1's expected values for P1: A and P1: B

P1: A - p(2)+(1-p)(-3)=-3+5p

P1: B - p(0)+(1-p)(3)=3-3p

P1 will not be able to take advantage of P2's mixed strategy if these two expected values are the same: -3+5p=3-3p. Solving, we get p=3/4. If P2 plays the mixed strategy ¾ A, ¼ B, P2 can assure that P1 wins, on average no more than ¾ unit per game, regardless of how P1 plays. This is known as the No Regret aspect. Also, if P2 plays the mixed strategy ¾ A, ¼ B, P2 is assured of winning, on average, at least -¾ units per game regardless of how P1 plays. This is no as the Security Level.

P1: A - ¾ (2) + ¼ (-3) = ¾

P1: B – ¾ (0) + ¼ (3) = ¾

Similar calculations are obtained for P2.

P2: A - p(2) + (1-p)(0) = 2p

P2: B – p(-3) + (1-p)(3) = 3-6p

Setting 2p=3-6p, we solve for p= 3/8.

P2: A - 3/8(2)+5/8(0)= ¾

P2: B – 3/8(-3)+5/8(3) = ¾

Reasoning as in the saddle point case, game theorists prescribe

- ¾ as the value of the game
- 3/8 A, 5/8 B as P1's optimal strategy
- ¾ A, ¼ B as P2's optimal strategy

Note that it is important to check for a saddle point before you use this method to find an optimal mixed strategy. If a game has a saddle point, then this method of mixed strategies will not produce optimal strategies.

The following theorem was proved by John von Neumann in 1928:

MINIMAX THEOREM. Every *m* x *n* matrix game has a solution. That is, there is a unique number *v,* called the value of the game, and there are optimal (pure of mixed) strategies for P1 and P2 such that

i) if P1 plays their optimal strategy, P1's expected payoff will be ≥ v, no matter what P2 does (Security Level), and
ii) if P2 plays their optimal strategy, P1's expected payoff will be ≤ v, no matter what Player 1 does (No Regret Aspect).

Consider a general 2 x 2 game:

		P2	
		A	B
P1	A	*a*	*b*
	B	*c*	*d*

Game 4.7

Game 4.7 will have a saddle point unless the two largest entries are diagonally opposite each other, so suppose the two largest entries are *a* and *d*. Suppose P2 plays A and B with probabilities p and (1-p). Then the value of p which will equalize P1's expectations for P1: A and Player 1: B is

p(a)+(1-p)b=pa+b-pb
p(c)+(1-p)d=pc+d-pd

Therefore: pa+b-pb=pc+d-pd
p(a-b-c+d)=d-b
p=(d-b)/(a-b-c+d)

Player 2's optimal strategy is
(d-b)/(a-b-c+d)A, (a-c)/(a-b-c+d)B

Similarly, suppose Player 1 plays A and B with probabilities p and (1-p). Then the value of p which will equalize Player 2's expectations for Player 2: A and Player 2: B is
p(a)+(1-p)c=pa+c-pc
p(b)+(1-p)d=pb+d-pd

Therefore: pa+c-pc=pb+d-pd
p(a-c-b+d)=d-c
p=(d-c)/(a-b-c+d)

Player 1's optimal strategy is
(d-c)/(a-b-c+d)A, (a-b)/(a-b-c+d)B

The value of the game is
v = (ad-bc) / (a-b-c+d)

4.5 Utilities

In the discussion so far, we have mostly assumed that the numerical payoffs in our game matrices are given. We have not paid much attention to where the numbers come from or exactly what they mean. It is time to consider more thoroughly the process of assigning numbers to outcomes, for the applicability of game theory to real situations rests on the assumption that this can be done in a reasonable way. Von Neumann and Morgenstern were very conscious of this dependence, and they began the *Theory of Games and Economic Behavior* by laying the groundwork for modern utility theory, the science of assigning numbers to outcomes in a way which reflects an actor's preferences.

One of the axioms states that numbers in the game matrix must be cardinal utilities and can be transformed by any positive linear function f(x)=ax+b, a>0 without changing the information they convey.

The invariance of cardinal utilities under positive linear functions means that some games which do not appear to be zero-sum are in fact equivalent to zero-sum games. Consider the two-person nonzero-sum game given by Game 4.8. If we transform P1's utilities by the positive linear function f(x) =½(x-17) we get Game 4.9 which is zero-sum

		P2	
		A	B
P1	A	(27,-5)	(17,0)
	B	(19,-1)	(23,-3)

Game 4.8

		P2	
		A	B
P1	A	(5,-5)	(0,0)
	B	(1,-1)	(3,-3)

Game 4.9

4.6 Application to tennis

Analyzing risk-taking strategies in tennis is complicated. There has been a tendency to analyze risk-taking on the serve more often than other shots. This seems reasonable as the serve is the first shot to be played and therefore simplifies the analysis by not having to consider previous shots in the rally. The majority of the literature analyzes the situation where the server is the only decision maker and therefore the optimal strategy will be a single strategy with certainty e.g. a player should always serve a 'typical' high risk first serve on both the first and second serves. Given there is an opponent receiving in tennis is makes sense to analyze risk taking on serve by also taking into account whether the receiver is expecting a low or high risk second serve (known more generally as game theory), the optimal strategy can be a mixed strategy e.g. a player should serve a 'typical' high risk first serve 20% of the time on the second serve and a 'typical' low risk second serve 80% of the time on the second serve.

Scenario a)

The following definitions are given to obtain a high and low risk serve for each player:

- A high risk serve is a 'typical' first serve by a player and calculations are obtained by a player's averaged percentage of points won on the first serve for a particular surface
- A low risk serve is a 'typical' second serve by a player and calculations are obtained by a player's averaged percentage of points won on the second serve for a particular surface

Let:
d_{hijs} = percentage of points won on high risk serves (unconditional) for player i, for when player i meets player j on surface s
d_{lijs} = percentage of points won on low risk serves (unconditional) for player i, for when player i meets player j on surface s

The following two serving strategies are defined:
Strategy 1 – high risk serve followed by a high risk serve
Strategy 2 – high risk serve followed by a low risk serve

Thus, player i should use Strategy 1 (two high risk serves) rather than Strategy 2 if $d_{hijs} > d_{lijs}$

An example of such a case is given between Andy Roddick (recognized as a 'strong' server) and Rafael Nadal (recognized as a 'strong' receiver), where the results from table 4.1 indicate that Roddick might be encouraged to serve high risk on both the first and second serve when playing Nadal on grass (since 0.535>0.512). However, he should use a high risk first serve and low risk second serve when playing Nadal on both hard court (since 0.528<0.551) and clay (since 0.364<0.458). This example illustrates the fact that it can be important for players to identify the match statistics for themselves and their opponents – specific to court surfaces.

	Andy Roddick			Rafael Nadal		
Statistic	Grass	Hard	Clay	Grass	Hard	Clay
d_{lijs}	0.512	0.551	0.458	0.582	0.571	0.608
d_{hijs}	0.535	0.528	0.364	0.510	0.495	0.546
matches	37	99	17	24	72	72

Table 4.1: Serving and receiving statistics for Andy Roddick and Rafael Nadal

Scenario b)

The model developed in scenario a) is now extended by taking into account strategies on whether the receiver is expecting a low or high risk second serve. From table 4.1, where Roddick is serving against Nadal on hard court, Roddick is expected to win 55.1% of points on the second serve when serving low risk on the second serve and expected to win 52.8% of points on the second serve when serving high risk on the second serve. Suppose these percentages are based on whether Nadal on the return of serve is expecting a high or low risk second serve. For example, if Roddick was serving a low risk second serve and Nadal was expecting a low risk second serve, then the percentage won on the second serve for Roddick would likely be less than 55.1%. This is represented in table 4.2 below in a game theory matrix with the following observation. If Nadal was expecting a low risk second serve 50% of the time and a high risk second serve 50% of the time (indifferent between strategies), then Roddick should always serve a low risk second serve since ½*0.53 + ½*0.57=0.55 and ½*0.55 + ½*0.51=0.53. These results are in agreement with the earlier model from scenario a) where decisions of the opponent were not taken into account.

Using standard game theory techniques to solve this two-person zero-sum game; gives mixed strategies for Roddick of 50% low risk serve, 50% high risk serve and for Nadal of 75% expecting a low risk serve, 25% expecting a high risk serve. The outcome of the game with both players' adopting these mixed strategies is such that Roddick will win 54% of points on the second serve. If either player deviated from these strategies then the other player could capitalize by changing strategies accordingly. For example, if Roddick changed strategies to 80% low risk serve, 20% high risk serve, then Nadal could choose the strategy of 100% expecting low risk serve, for an outcome of Roddick to win 0.53*0.8+0.55*0.2=53.4% of points on the second serve.

		Nadal	
		expecting low risk serve	expecting high risk serve
Roddick	low risk serve	0.53	0.57
	high risk serve	0.55	0.51

Table 4.2: Game theory matrix of how much risk to take on the second serve in tennis

The model developed in scenario a) is now extended to include the 'importance' of points. The results obtained also extend to the model developed in scenario b).

The following result follows from Klaassen and Magnus (2001), where it was established that a server's probability of winning a point decreases with the more 'important' points.

Player i should use Strategy 1 (two high risk serves) rather than Strategy 2 if $d_{hijs} > d^{\wedge}_{lijs}$. The superscript ^ is used as the server's probability of winning a point on a low risk serve is now conditional on the 'importance' of the point.

This is evidence to suggest that the server would be encouraged to take more risk on the second serve on the more 'important' points as a mixed strategy approach.

4.7 Application to hierarchical games

Hierarchical games typically apply to racket sports, such as (points, games, set, match) in tennis and (points, games, match) in table tennis. However hierarchical games can also exist in warfare (skirmishes, battles, campaigns, war). Hence, we will analyze the situation with two players where an increase in effort by each player is applied at sets within a tennis match, games within a table tennis match, or campaigns within a war. For notational purposes we will call this increase in effort a "set" within a "match" as typically applies to tennis.

We now model the situation where both players can apply an increase in effort, which is represented by a two-person zero-sum game. For a best-of-3 set match, either player can apply an increase in effort at the first, second or third set played, resulting in a total of 9 possibilities. An increase in effort by ε at a set played from player A, results in increasing p to $p + \varepsilon$ ($p + \varepsilon < 1$), and an increase in effort by α at a set played from player B, results in decreasing p to $p - \alpha$ ($p - \alpha > 0$), where p represents the probability of player A winning a set. For the time being, it is assumed that both players must decide before the match has begun, on which set played that an increase is to be applied, and cannot change this choice throughout the match. Table 4.3 represents the probabilities of player A winning the match when an increase in effort is applied at the various sets played, where I_A and I_B represent an increase in effort at a set played by players A and B respectively. Notice that when both players apply an increase in effort on the same set played, the probability of player A winning the match is the same. Similarly, when both players apply an increase in effort on different sets played, the probability of player A winning the match is the same. When:

$p^2(3 - 2p) + 2p(1 - p)(\varepsilon - \alpha) + \alpha\varepsilon(2p - 1) > p^2(3-2p) + 2p(1- p)(\varepsilon - \alpha)$
$\rightarrow \alpha\varepsilon(2p-1) > 0$
$\rightarrow p > ½$

Similarly, when:
$p^2(3 - 2p) + 2p(1 - p)(\varepsilon - \alpha) + \alpha\varepsilon(2p - 1) < p^2(3-2p) + 2p(1- p)(\varepsilon - \alpha)$
$\rightarrow \alpha\varepsilon(2p-1) < 0$
$\rightarrow p < ½$

I_A	I_B	Probability of player A winning
0	0	$p^2(3\text{-}2p) + 2p(1\text{-} p)(\varepsilon - \alpha)$
1	0	$p^2(3 - 2p) + 2p(1 - p)(\varepsilon\text{-} \alpha) + \alpha\varepsilon(2p\text{-} 1)$
0	1	$p^2(3 - 2p) + 2p(1 - p)(\varepsilon\text{-} \alpha) + \alpha\varepsilon(2p\text{-} 1)$
1	1	$p^2(3\text{-}2p) + 2p(1\text{-} p)(\varepsilon - \alpha)$
2	0	$p^2(3 - 2p) + 2p(1 - p)(\varepsilon\text{-} \alpha) + \alpha\varepsilon(2p\text{-} 1)$
0	2	$p^2(3 - 2p) + 2p(1 - p)(\varepsilon\text{-} \alpha) + \alpha\varepsilon(2p\text{-} 1)$
1	2	$p^2(3 - 2p) + 2p(1 - p)(\varepsilon\text{-} \alpha) + \alpha\varepsilon(2p\text{-} 1)$
2	1	$p^2(3 - 2p) + 2p(1 - p)(\varepsilon\text{-} \alpha) + \alpha\varepsilon(2p\text{-} 1)$
2	2	$p^2(3\text{-}2p) + 2p(1\text{-} p)(\varepsilon - \alpha)$

Table 4.3: Probability of player A winning the match when an increase in effort is applied by both players at a set played in a match.

The increase in probability of winning for the better player when an increase in effort for both players is applied on different sets, is a result of the variability about the overall mean. Let $X = p^2(3\text{-}2p) + 2p(1\text{-} p)(\varepsilon - \alpha)$and $Y = p^2(3 - 2p) + 2p(1 - p)(\varepsilon\text{-} \alpha) + \alpha\varepsilon(2p\text{-} 1)$. Let strategy *Ki* (*K:* {A,B}*, i :* {1*,* 2*,* 3}) refer to player *K* applying an increase in effort at *i* sets played. The game theory matrix is represented by:

			Player B	
		B1	B2	B3
	A1	X	Y	Y
Player A	A2	Y	X	Y
	A3	Y	Y	X

This matrix can easily be solved and the results indicate that players A and B should apply mixed strategies of A: (⅓, ⅓,⅓) and B: (⅓, ⅓,⅓). The value (v) of the game is then ⅓X + ⅔Y. For example, if $p = 0.25$, $\varepsilon = \alpha = 0.1$, then $X = 0.15625$; $Y = 0.15125$ and $v = 0.1529$.

Suppose either player can now alter their strategies as the match is in progress. When should either player apply an increase in effort to optimize the usage of their available increase?

Consider the following analysis. Suppose at the start of the match, player A decides to apply an increase in effort at the first set played with a score line of (0,0), and player B decides to apply an increase in effort at the third set played with a score line (1,1). After the first set has been played, player B now has a decision to make on whether to stay with the initial strategy, by applying an increase in effort at the third set, or change strategies and apply an increase in effort at the second set played. As previously calculated in the above section, player B has the same probability of winning the match by applying an increase at the second or thirds sets played. Therefore, player B could change their initial strategy by applying an increase in effort at the second set, and have optimized the usage of their available increase. Similarly, if player B decides at the start of the match to apply an increase in effort at the first set played, and player A decides to apply an increase in effort at the third set played, then player A could change their initial strategy by applying an increase in effort at the second set, and have optimized the usage of their available increase. This analysis is summarized as follows:

1. Both players are to apply an increase in effort at the first set played with probability of ⅓ .
2. If one player applies an increase in effort at the first set played, then the other player can decide to apply an increase in effort at either the second or third sets played. If neither player increased their effort at the first set played, then both players are to apply an increase in effort at the second set played with probability of ½
3. If the match reaches (1,1) and neither player has applied their increase in effort, then the increase in effort by both players must be applied at this state of the match.

Note that the mixed strategies of A: (⅓, ⅓,⅓) and B: (⅓, ⅓,⅓) still gives an optimal solution.

A best-of-3 set match is extended to a best-of-*N* set match in the case *N* odd by the following conjecture:

Suppose both players can apply one increase in effort in a best-of-*N* set match (*N*: odd integer). An increase in effort by ε at a set played from player A, results in increasing p to $p + \varepsilon$ ($p + \varepsilon < 1$), and an increase in effort by α at a set played from player B, results in decreasing p to $p - \alpha$ ($p - \alpha > 0$), where p represents the probability of player A winning a set. Then an optimal strategy for both players is to decide at the start of the match to apply the increase in effort with equal probability at *N* sets played, where the probability of applying the increase in effort at a set is given by $1/N$. The value of the game is given by $1/N\, X + (N\text{-}1)/N\, Y$, where X represents the probability of player A winning the match when an increase in effort by each player is applied at the same set played, and Y represents the probability of player A winning the match when an increase in effort by each player is applied at different sets played.

4.8 Application to poker

In matrix games we have assumed that the players make their choice of strategy simultaneously, without knowledge of what the other player is choosing. We will consider a method of modelling decisions sequentially by a game tree. We will find that, perhaps surprisingly, this new model can always be reduced to our model of a matrix game.

Consider the following radically simplified version of a game of poker. Each of two players, Player 1 and Player 2, puts $1 into the pot as “ante”. Each is then dealt a hand, which consists of one card, from a large deck which consists only of aces and kings. Player 2 must decide whether to bet $2 or to drop. If he drops, Player 1 wins the pot. If Player 2 bets, Player 1 must decide whether to call by matching Player 2’s bet, or to fold. If Player 1 folds, Player 2 wins the pot. If Player 1 calls, the players compare their hands and the higher card wins the pot. If the hands tie, the pot is split equally.

If we label the rows and columns of a matrix by the possible strategies of Player 1 and Player 2, and enter the corresponding expected payoffs in the matrix, we have a matrix game which corresponds to the game originally presented as a tree. Thus, every game tree can be reduced to a game matrix.

To illustrate, suppose that in the poker game Player 2 chooses the strategy of betting only if they have an ace, and Player 1 chooses the strategy of calling only if they have an ace. We can then calculate the outcome for each possible CHANCE move:

Probability	Player 1 hand, Player 2 hand	Outcome	Payoff to Player 1
¼	A, A	Player 2 bets, Player 1 calls, tie	0
¼	A, K	Player 2 drops	+1
¼	K, A	Player 2 bets, Player 1 folds	-1
¼	K, K	Player 2 drops	+1

The expected payoff to Player 1 for these choices of strategies is (¼)(0) + (¼)(1) + (¼)(-1) + (¼)(1) = ¼. If we do the corresponding calculation for the other fifteen possible pairs of strategies and enter the results in a matrix, we get a matrix game as given by Game 4.10.

			Player 2		
		always	A only	K only	never
	always	0	-¼	5/4	1
Player 1	A only	~~¼~~	~~¼~~	1	1
	K only	-5/4	-½	¼	1
	never	-1	0	0	1

Game 4.10

From Game 4.10, we see that the game has two saddle points as indicated by a strikethrough, with value ¼. Player 2 should always bet with an ace, and may bet with a king. Player 1 should call only with an ace. Player 1 can expect to win an average of 25 cents per game.

4.9 References

Straffin P (1993), Game Theory and Strategy. The Mathematical Association of America, Washington.

Chapter 5
Two-person zero-sum games with risk

5.1 Introduction

The origins of game theory extend back to 0-500AD where the so-called marriage contract problem is discussed in the Talmud. In 1713 James Waldegrave provided the first known Minimax mixed strategy solution to a two-person game, but expressed concern that a mixed strategy "does not seem to be in the usual rules of play" of games of chance. Perhaps the "official" beginning of game theory was the 1944 book *Theory of Games and Economic Behavior* (Von Neumann and Morgenstern, 1944). Two-person zero-sum game theory is covered as well as the framework for modern axiomatic utility theory by assigning numbers to outcomes in a way that reflect an actor's preference. It was also the account of axiomatic utility theory given there that led to its wide spread adoption within economics and law. One of the axioms states that numbers in the game matrix must be cardinal utilities and can be transformed by any positive linear function f(x)=ax+b, a>0 without changing the information they convey.

Consider Game 5.1 and Game 5.2, where Game 5.2 is a linear transformation of Game 5.1 by adding b=4 to all utility values given by Game 5.1. The strategies given by the Minimax Theorem are P1: A=0.2, B=0.8 and P2: A=0.7, B=0.3. P1 may choose to always play strategy B in Game 5.1 to guarantee at least a positive payout of +2. If P1 was to use the strategy as given by the Minimax Theorem in Game 5.1, then he could end up with a negative payout of -3, even though the expected payout of 2.6 is positive. However, P1 may choose to play the strategies given by the Minimax Theorem in Game 5.2, since is always guaranteed a positive payout of +1. This example contradicts the Von Neumann-Morgenstern linearity axiom and indicates that whilst a player may choose to maximize expectations, a player (presumably the player with an expected positive payout) may be concerned about obtaining a negative payout or their maximum possible loss (MPL) for the game. This is likely to be more of a concern when the MPL is a negative payout and to reduce the MPL occurring, a player may choose strategies accordingly. In effect, the maximal expectation is reduced by minimizing the probability of obtaining the MPL i.e. using risk-averse strategies. Risk-averse strategies have foundations in gambling theory when a favourable game exists. Friedman (1980) and Schlesinger (2004) outline risk-averse strategies in blackjack where the objective is to minimize the probability of losing one's bankroll.

		P2	
		A	B
P1	A	5	-3
	B	2	4

Game 5.1

		P2	
		A	B
P1	A	9	1
	B	6	8

Game 5.2

5.2 Existing strategies

Consider the following 2 x 2 game (call it Game 5.3)

		P2	
		A	B
P1	A	2/3	-1
	B	-1/3	1

Game 5.3

The most common solution to this type of game is using the Minimax Theorem to obtain player strategies. This gives mixed strategies as P1: 4/9A, 5/9B and P2: 2/3A, 1/3B. The value of the game is 1/9. Table 5.1 represents Game 5.3 in a form where both players apply strategies from the Minimax Theorem. Outcome AA refers to P1 using strategy A and P2 using strategy A. Outcomes AB, BA and BB are obtained similarly. The value of the game is given as 1/9 as expected. This representation is typically given for a casino game.

Outcome	Payout	Probability	Expected Payout
AA	2/3	4/9 * 2/3 = 8/27	2/3*8/27 = 16/81
AB	-1	4/9 * 1/3 = 4/27	-1*4/27 = -4/27
BA	-1/3	5/9 * 2/3 = 10/27	-1/3*10/27 = -10/81
BB	1	5/9 * 1/3 = 5/27	1 * 5/27 = 5/27
		1	**1/9**

Table 5.1: Probabilities and expected payouts for Game 3 with both players using strategies under the Minimax Theorem

Table 5.2 is an extension of Table 20.1 which is used to obtain moments, which can then be used to obtain cumulants and common distributional characteristics as follows:
Mean = 0.111
Standard Deviation = 0.703
Coefficient of Variation = 6.325
Coefficient of Skewness = -0.158
Coefficient of Excess Kurtosis = -1.425

These distributional characteristics (other than the mean) provide extra information to the possible payouts in a two-person zero-sum game.

Outcome	Payout	Probability	1st Moment	2nd Moment	3rd Moment	4th Moment
AA	2/3	8/27	0.198	0.132	0.088	0.059
AB	-1	4/27	-0.148	0.148	-0.148	0.148
BA	-1/3	10/27	-0.123	0.041	-0.014	0.005
BB	1	5/27	0.185	0.185	0.185	0.185
			0.111	**0.506**	**0.111**	**0.396**

Table 5.2: The first four moments for Game 24.3 with both players using strategies under the Minimax Theorem

5.3 Other strategies

Commonly used strategies other than the strategies determined by the Minimax Theorem are as follows as documented in Straffin (1993): From Game 5.3, Suppose P1 recognizes that P2 will play mixed strategies as 0.8A, 0.2B. Then using the Expected Value Principle, P1 would use the fixed strategy of A since the reward for the game is 1/3. Players may want to be cautious and Wald's method for P1 is to write down the minimum entry in each row and choose the row with the largest minimum. This would mean P1 would use a fixed strategy of B. An analog for P2 would be to use the fixed strategy of A. As Wald's *maximin* strategy is looking at the worst that will happen, the corresponding principle for optimists would be the *maximax* principle. This would mean P1 would use a fixed strategy of B and P2 would use a fixed strategy of B. Hurwicz combined these two approaches by choosing a "coefficient of optimism" α between 0 and 1. For each row compute α(row maximum) + (1- α)(row minimum). Choose the row for which this weighted average is the highest. For example, suppose we choose α = 0.8, then A: 0.8 (2/3) + 0.2 (-1) = 1/3 and B: 0.8 (-1/3) + 0.2 (1) = -1/15. Hence P1 would choose strategy A. Savage's method involves regret by writing down the largest entry in each row. Choose the row for which this largest entry is smallest. This would mean P1 would choose the fixed strategy of A and P2 would choose the fixed strategy of A.

As mentioned in section 4.5, Von Neumann and Morgenstern (1944) developed the framework for modern utility theory by assigning numbers to outcomes in a way that reflect an actor's preference. It is stated that for a mixed strategy game solution to be meaningful, the numbers in the game matrix must be cardinal utilities and can be transformed by any positive linear function f(x)=ax+b, a>0 without changing the information they convey. All the strategies above and the strategies determined by the Minimax Theorem conform to this property. However, this property does not take into account risk and whether a player (presumably the player with an expected positive payout) would reduce the expected payout in order to reduce risk; such as minimize the probability of obtaining the maximum possible loss. Section 5,4 will outline methods to reduce risk for the favourable player in a two-person zero-sum game.

5.4 Risk-averse strategies

		P2	
		A	B
P1	A	*a*	*b*
	B	*c*	*d*

Game 5.4

Consider a general 2 x 2 zero-sum game as given by Game 5.4. To apply mixed strategies using the Minimax Theorem we will assign $d>a>c>b$, $c>0$ and $b<0$. This gives the optimal mixed strategy for P1 as (d-c)/(a-b-c+d)A, (a-b)/(a-b-c+d)B. The value of the game is calculated as v = (ad-bc) / (a-b-c+d). Suppose P1 is the favourable player such that $v>0$. P1 can deviate from Minimax strategies by either increasing strategy A (which decreases strategy B) or increasing strategy B (which decreases strategy A). By increasing strategy A, the maximum possible loss (MPL) consisting of payout b will increase and similarly by increasing strategy B, the MPL will decrease. Therefore, the deviation from Minimax strategies for P1 can be thought of in terms of the increase or decrease in the MPL. Since P1 can guarantee an expected positive payout by playing Minimax strategies, we will assume that any strategy that P1 adopts will give an expected positive payout regardless of the strategies used by P2.

Table 5.3 represents the distributional characteristics for Game 5.3 with P2 using Minimax strategies in all columns and P1 using Minimax strategies (column 2), strategies that decrease the MPL (column 3) and strategies that increase the MPL (column 4). It is observed from column 3 that the standard deviation, coefficient of variation and coefficients of skewness and excess kurtosis are reduced when compared to column 2. It is observed from column 4 that the standard deviation, coefficient of variation and coefficients of skewness and excess kurtosis are increased when compared to column 2.

Distributional Characteristics	P1: 4/9A, 5/9B P2: 2/3A, 1/3B	P1: 0.4A, 0.6B P2: 2/3A, 1/3B	P1: 0.48A, 0.52B P2: 2/3A, 1/3B
Mean	0.111	0.111	0.111
Standard Deviation	0.703	0.696	0.708
Coefficient of Variation	6.325	6.261	6.375
Coefficient of Skewness	-0.158	-0.095	-0.206
Coefficient of Excess Kurtosis	-1.425	-1.424	-1.427
Maximum Possible Loss	0.148	0.133	0.160
Overall Loss	0.519	0.533	0.507

Table 5.3: Distributional characteristics of Game 24.3 for various strategies used by P1

Based on the above result, Definition 5.1 describes risk-averse strategies for a 2 x 2 zero-sum game.

Definition 5.1: Consider a 2 x 2 zero-sum game where at least one of the payouts is positive and at least one of the payouts is negative. The value *v* of the game under the Minimax Theorem is either positive, negative or zero. Risk-averse strategies can be obtained when *v is* positive such that the expected payout for P1 is positive regardless of the strategies used by

P2, and the probability of obtaining the maximum possible loss (MPL) for P1 in the game is reduced when compared to the strategies under the Minimax Theorem. Risk-averse strategies for P2 when *v* is negative follow.

Using Definition 5.1, the risk-averse solution for P1 to Game 5.3 is obtained as P1: $1/3 < A \leq 4/9$, $5/9 \leq B < 2/3$. These calculations were obtained by noting that P1 always obtains an expected positive payout regardless of the strategies used by P2.

Example: Using Game 5.3, suppose P1 is restricting the probability of the -1 payout to be 0.12 (given P2 uses mixed strategies obtained from the Minimax Theorem). Then 0.12 / 1/3 = 0.36 and P1 should use the risk-averse strategy of P1: 0.36A, 0.64B. If P2 identified that P1 was deviating from the Minimax Theorem, then P2 could use strategy A for an expected payout for P2 of -0.027.

This idea of reducing the MPL is extended to two-person zero sum games.

Definition 5.2: Consider a two-person zero-sum game where at least one of the payouts is positive and at least one of the payouts is negative. The value *v* of the game under the Minimax Theorem is either positive, negative or zero. Risk-averse strategies can be obtained when *v is* positive such that the expected payout for P1 is positive regardless of the strategies used by P2, and the probability of obtaining the maximum possible loss (MPL) for P1 in the game is reduced when compared to the strategies under the Minimax Theorem. Risk-averse strategies for P2 when *v* is negative follow.

5.5 Risk of ruin

A common problem that often arises in gambling is obtaining the probability of losing one's entire bankroll given a favourable game. The following recursive solution (which assumes independent trials) was derived by Evgeny Sorokin and posted on Arnold Snyder's Blackjack Forum Online
http://www.blackjackforumonline.com/content/VPRoR.htm.

The equation is given as $R(1) = E[p_i * R(1)^{Z_i}]$

where:
R(1) is the risk of losing a 1 unit bankroll
Z_i is the return payoff for outcome i
p_i is the associated probability for Z_i

In the context of Game 5.3, suppose P1 has a bankroll of 3 units. With P1 and P2 playing strategies under the Minimax Theorem, the risk of ruin for P1 is obtained as 26.0877%. Suppose P1 increased strategy B to 0.613, then the risk of ruin (with P2 playing strategies under the Minimax theorem) is reduced to 24.87227%. If P2 then increased strategy A (to reduce the expected payout for P1), the risk of ruin for P1 would only decrease as shown in table 5.4.

Therefore, the risk of ruin is reduced by P1 playing strategy B with probability 0.613, regardless of the strategies used by P2.

	P1: A = 0.387, B =0.613	
Probability	Risk of Ruin for P1	Expected Payout
P2: A=2/3	24.87227%	0.11111
P2: A=0.667	24.87197%	0.11105
P2: A=0.668	24.87107%	0.11088
P2: A=0.70	24.84051%	0.10537

Table 5.4: Risk of ruin and expected payouts for Game 5.3 for various strategies used by P1 and P2

5.6 Kelly Equilibrium

The Kelly Criterion is applied to Game 5.3 to demonstrate why the favourable player may consider risk-averse strategies. The value of b^* with both P1 and P2 using Minimax strategies is obtained as 0.222. Therefore P1 (the favourable player) should wager an amount of 0.222 * current bankroll to maximize the long-term growth of the bank. Since the amount that P1 can bet is fixed at 1 unit, the decision on whether to apply Minimax strategies or risk-averse strategies can depend on the size of the bankroll. Using Solver in Excel, Table 5.5 shows that P1 would need a bankroll of at least 4.51 units to avoid over betting by using Minimax strategies. By using risk-averse strategies, P1's bankroll can be less than 4.51 as given in Table 5.5.

Strategy	Bankroll (units)
P1: A=4/9, B=5/9	4.51
P1: A=0.4, B=0.6	4.38
P1: A=0.35, B=0.65	4.23

Table 5.5: Bankroll requirements when using strategies under the Kelly Criterion

Suppose the payouts for each player in Game 5.3 was to be determined by an outside arbitrator. One obvious method is simply to use the value of the game given by the Minimax theorem. For Game 5.3 this would be 1/9 to P1. However, it would appear to be more of an incentive for the favourable player to have the game determined by arbitration rather than play the game simultaneously, as they run the risk of being at a loss even though the expected outcome is positive. For example, from table 5.3, if both players are playing Minimax strategies, there is a 0.148 probability of ending up with the MPL of -1 on any trial and a 0.519 probability of ending up with any loss on any trial. The favourable player can of course reduce the MPL by playing risk-averse strategies, but as a consequence could reduce the expected amount if the other player changed strategies accordingly. This illustration suggests that the arbitration amount to the favourable player should be less than the expected amount as given by the Nash Equilibrium (Minimax theorem).

Suppose the game given by Table 5.1 was a casino game and the player had a finite bank. If the player bet the same amount on each trial then the expected profit on each trial would be 0.11111 for a unit bet. If the player had a bankroll of 3 units, then the chance of ruin as given above is 26.0877%. The expected profit on each trial differs under the Kelly Criterion method since the player's bankroll changes each trial according to the wins and losses, and hence we will adopt an averaged expected profit notation. If the player applied the Kelly Criterion with a bankroll of 3 units, then the averaged expected profit on each trial would be 0.11111 * 0.2217 = 0.0246, and the chance of ruin would approach zero. Given the Kelly Criterion maximizes the long-term growth of the bank, this would appear to be a reasonable method in a favourable gambling context, and shows that the averaged expected amount of profit is less than the amount given by fixed betting on each trial. Based on this reasoning an arbitration value could be determined by the averaged expected profit as given under the Kelly Criterion. For Game 5.3 this value is given as 0.0246. This analysis leads to the Kelly Equilibrium in a 2 x 2 zero-sum game as given by Definition 5.3.

Definition 5.3: Consider a 2 x 2 zero-sum game where at least one of the payouts is positive, at least one of the payouts is negative and the value of the game v is positive. The Kelly Equilibrium can be obtained by v.b*, where b* is given by the Kelly betting fraction with both players adopting Minimax strategies and P1 then uses risk-averse strategies such that the expected payout is v.b* with P2 playing strategies under the Expected Value Principle. Alternatively, the Kelly Equilibrium and strategies for P1 and P2 when *v* is negative follow.

For Game 5.3, the Kelly Equilibrium is given by the strategies of P1: A=0.3579, B=0.6421 and P2: A=1, B=0 with a payout of 0.0246. Note how these strategies using the Kelly Equilibrium are within the risk-averse region of P1: 1/3<A≤4/9, 5/9≤B<2/3. Table 5.6 summarizes the Nash and Kelly Equilibrium.

	Game 3
Nash Equilibrium	P1: A=0.4444, B=0.5556 P2: A=0.6667, B=0.3333 Payout: 0.1111
Kelly Equilibrium	P1: A=0.3579, B=0.6421 P2: A=1, B=0 Payout: 0.0246

Table 5.6: The Nash Equilibrium and Kelly Equilibrium for Game 5.3

This analysis given leads to the Kelly Equilibrium in a two-person zero-sum game as given by Definition 5.4.

Definition 5.4: Consider a two-person zero-sum game where at least one of the payouts is positive, at least one of the payouts is negative and the value of the game v is positive. The Kelly Equilibrium can be obtained by v.b*, where b is given by the Kelly betting fraction with both players adopting Minimax strategies and P1 then uses risk-averse strategies such that the expected payout is v.b* with P2 playing strategies under the Expected Value Principle. Alternatively, the Kelly Equilibrium and strategies for P1 and P2 when *v* is negative follow.

Software to determine the 'Kelly Equilibrium' in 2 × 2 zero-sum games can be obtained here: http://strategicgames.com.au/conflicts.xlsx

5.7 References

Friedman, J. (1980) Risk Averse Playing Strategies in the Game of Blackjack. ORSA Conference at the University of North Carolina at Chapel Hill.

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