Duration of a game of tennis - using backward recursion

Let $\mu(Y(a,b))$ represent the mean number of points remaining in the game at point score (a,b). Let p represent the probability of the server winning a point and q=1-p.
The following derivation is used to obtain the recurrence formula.

Consider the random variable $X(a,b)$ of the total number of points played in a game at point score (a,b). If the game has not reached its completion then the next point must be contested, and one of two results must occur. If the server wins the point then the score progresses to $(a + 1, b)$; otherwise the score progresses to $(a, b + 1)$. It follows that:
$X(a,b)=X(a+1,b)$ with probability p, and
$X(a,b)=X(a,b+1)$ with probability q

Taking expectations we obtain a backwards recurrence formula
$E(X(a,b)) = pE(X(a+1,b)) + qE(X(a,b+1))$

The mixture law above applies at each and every score before the game is completed.

Consider the random variable $Y(a,b)$ of the number of points remaining in a game at point score (a,b) for the server. Then
$X(a,b)=a+b+Y(a, b)$ for all $a \geq 0, b \geq 0$

As the score progresses we now have
$Y(a,b) = 1 + Y(a + 1, b)$ with probability p, and
$Y(a, b) = 1 + Y(a, b + 1)$ with probability q.

Taking expectations we obtain a backwards recurrence formula
$E(Y(a,b)) = 1 + pE(Y(a + 1, b)) + qE(Y(a, b + 1))$

The mixture law above applies at each and every score before the game is completed.

Therefore the recurrence formula for the mean number of points remaining in a game at point score (a,b) is:
$\mu(Y(a, b)) = 1 + p\mu(Y(a + 1, b)) + q\mu(Y(a, b + 1))$

The boundary value $\mu(Y(3,3))$ is obtained as follows.

Let $M_{Y(a,b)}(t)$ represent the moment generating function for the number of points remaining in a game from point score (a,b). Therefore: $M_{Y(3,3)}(t)=(p^2+q^2)e^{2t}/(1-2pqe^{2t})$

The first moment $E(Y(a,b))$ is obtained as $M^{(1)}{}_{Y(3,3)}(0)=2(p^2+q^2)/(1-2pq)^2 = 2/(p^2+q^2)$.
Therefore $\mu(Y(3,3)) = E(Y(a,b)) = 2/(p^2+q^2)$

Therefore the boundary values are obtained as
$\mu(Y(a,b)) = 0$, if b=4 and $a \leq 2$; a=4 and $b \leq 2$
$\mu(Y(3,3)) = 2/(p^2+q^2)$

Table 1 lists the mean number of points remaining in a game from point score (a,b) with p=0.60. It indicates that the mean number of points remaining in such a game is 6.5.

		B score				
		0	15	30	40	game
A score	0	6.5	6.0	4.8	2.8	0
	15	5.2	5.0	4.5	3.0	0
	30	3.6	3.7	3.8	3.3	0
	40	1.8	2.0	2.5	3.8	
	game	0	0	0		

Table 1: The mean number of points remaining in a game from various score lines with p=0.60

The following analysis is used to obtain $\sigma^2(Y(a,b))$, the variance of the number of points remaining in the game at point score (a,b).

Clarke and Norman[1] used recurrence relations to calculate probabilities of winning, mean and variance of lengths to squash. In particular they showed for a random variable Z which takes the value Z_1 with probability Π and the value Z_2 with probability 1 - Π, that
$E(Z)= \Pi E(Z_1)+(1-\Pi)E(Z_2)$
$\sigma^2(Z)= \Pi\, \sigma^2(Z_1)+(1-\Pi)\, \sigma^2(Z_2)+ \Pi(1- \Pi)(E(Z_1)- E(Z_2))^2$

Since the equation representing $E(Z)$ is in the same format as $E(X(a,b))$, then it follows that $\sigma^2(X(a,b))$, the variance of the total number of points played in the game at point score (a, b) is given by:

$\sigma^2(X(a,b))=p\sigma^2(X(a+1,b)+q\sigma^2(X(a,b+1)+pq(\mu(X(a+1,b))- \mu(X(a,b+1)))^2$

Let $\mu(X(a,b))$ represent the mean of the total number of points played in a game at point score (a,b).

Given $X(a,b)=a+b+Y(a,b)$, with a and b fixed, it follows that:
$\mu(X(a,b))$
$= \mu(a+b+Y(a,b))$
$= \mu(Y(a,b))+a+b$

Let $\sigma^2(X(a,b))$=represent the variance of the total number of points played in a game at point score (a,b).

Since $E(X(a,b))= \mu(X(a,b))$, it follows that:
$E(X(a,b))=a+b+ E(Y(a,b))$

$E(X^2(a,b))=(a+b)^2+2(a+b)\, E(Y(a,b))+ E(Y^2(a,b))$
$\sigma^2(X(a,b))= E(X^2(a,b))- E(X(a,b))^2= \sigma^2(Y(a,b))$

Therefore:
$\sigma^2(Y(a,b))= p\sigma^2(Y(a+1,b)+q\sigma^2(Y(a,b+1)+pq(\mu(Y(a+1,b))- \mu(Y(a,b+1)))^2$

The boundary value $\sigma^2(Y(3,3))$ is obtained as follows.

Using the analysis above to obtain the moment generating function for the number of points remaining in a game from point score (a,b), the second moment $E(Y^2(a,b))$ is obtained as $M^{(2)}_{Y(3,3)}(0) = 4(1+2pq)/(p^2+q^2)^2$

Therefore $\sigma^2(Y(3,3))= E(Y^2(a,b))- E(Y(a,b))^2 = 8pq/(p^2+q^2)^2$

Therefore the boundary values are obtained as
$\sigma^2(Y(a,b))=0$, if $b=4$ and $a\leq 2$; $a=4$ and $b\leq 2$
$\sigma^2(Y(3,3)) = 8pq/(p^2+q^2)^2$

Table 2 lists the variance of the number of points remaining in a game from point score (a,b) with $p = 0.60$. It indicates that the variance of the number of points remaining in such a game is 6.7.

		B score				
		0	15	30	40	game
A score	0	6.7	7.2	7.7	6.5	0
	15	6.2	6.7	7.4	7.3	0
	30	4.9	6.1	7.1	7.8	0
	40	2.6	4.1	6.4	7.1	
	game	0	0	0		

Table 2: The variance of the number of points remaining in a game from various score lines with p=0.60

Reference

[1]Clarke, S and Norman, J. Comparison of North American and international squash scoring systems: analytical results, Research Quarterly 50(4) (1979), 723-728.