Winning a game of tennis: using forward recursion

In[1], backward recursion formulas were developed to obtain the probabilities of winning a game conditional on the point score. Here we obtain spreadsheet calculations using forward recursion formulas for the probabilities of reaching a point score in a game (i.e. winning the game to 0). These calculations on reaching score lines can also be used to obtain probabilities of winning a game from the outset. The basic idea is that we start at the beginning of the game with an initial value and see what happens as the game progresses.

Let $N(g,h)$ be the probability of reaching a point score (g,h) in a game from the outset where g and h represent the projected point score for the server and receiver respectively. Let p represent the probability of the server winning a point and $q=1-p$.

The initial value is $N(0,0)=1$

The forward recursion formulas are:

$N(g,h)=pN(g-1,h)$, if $g=4$ and $0 \leq h \leq 2$; $h=0$ and $1 \leq g \leq 4$; $g \geq 3$, $h \geq 3$ and $g=h+1$; $g \geq 3$, $h \geq 3$ and $g=h+2$
$N(g,h)=qN(g,h-1)$, if $h=4$ and $0 \leq g \leq 2$; $g=0$ and $1 \leq h \leq 4$; $g \geq 3$, $h \geq 3$ and $h=g+1$; $g \geq 3$, $h \geq 3$ and $h=g+2$
$N(g,h)=pN(g-1,h)+qN(g,h-1)$, if $1 \leq g \leq 3$ and $1 \leq h \leq 3$; $g \geq 4$, $h \geq 4$ and $g = h$

Table 1 lists the probability of reaching various score lines in a game from the outset with $p = 0.60$. It indicates that the probability of reaching deuce in such a game is 0.276.

		B score				
		0	15	30	40	game
	0	1	0.400	0.160	0.064	0.026
	15	0.600	0.480	0.288	0.154	0.061
A score	30	0.360	0.432	0.346	0.230	0.092
	40	0.216	0.346	0.346	0.276	
	game	0.130	0.207	0.207		

Table 1: The probability of reaching various score lines in a game from the outset with $p=0.60$

The probability of player A winning the game on serve from the outset can be obtained from:
$N(4,0)+N(4,1)+N(4,2)+N(3,3)P(3,3) = 0.130+0.207+0.207+0.276 \times 0.692 = 0.736$ when $p = 0.60$.

Let $P(a,b)$ represent the probability of the server winning the game from point score (a,b), where a and b are score lines for the server and receiver respectively.

Note that we have used here the value of $P(3,3)=0.692$ obtained from a formula given in[2]

After the point score of (3,3) has been reached the recursive formulas tell us that

$N(4,3) = pN(3,3)$,
$N(3,4) = qN(3,3)$, and
$N(4,4) = pN(3,4) + qN(4,3)$.

These results can be summarized as
$N(4,4) = 2pqN(3,3)$.

This argument can be continued to establish that
$N(n,n) = (2pq)^{n-3}N(3,3)$ for all $n \geq 3$.

Now to win a game after any deuce either player must win two consecutive points. This occurs with probability p^2+q^2. Thus the completion of the game after the first deuce whilst serving can be expressed in two equivalent ways:
$N(3, 3) = (p^2 + q^2)(N(3,3) + N(4,4) + N(5, 5) + \ldots\ldots)$
$= (p^2 + q^2)N(3,3)(1 + 2pq + (2pq)^2 + \ldots\ldots)$

If we substitute $r = 2pq$, then $0 < r \leq 1/2$ when $0 < p < 1$ and $q = 1 - p$ whilst $p^2 + q^2 = 1 - r$. By cancelling the term $N(3,3)$, which is not zero, we find that $1 = (1 - r)(1 + r + r^2 + \ldots)$

This result can be expressed as the sum of an infinite geometric series by the following theorem:

Theorem 1.1. $1+r+r^2+\ldots = 1/(1-r)$ *for* $0<r<1/2$.

Reference

[1]Barnett T. Winning a game of tennis: using backward recursion. To appear in JMST
[2]Barnett T. Winning a game of tennis: using the Binomial theorem. To appear in JMST