Winning a game of tennis: using Markov Chain theory

A Markov Chain, named for Andrey Markov, is a mathematical system that undergoes transitions from one state to another (from a finite or countable number of possible states) in a chainlike manner. It is a random process characterized as memoryless (i.e. exhibiting the Markov property): the next state depends only on the current state and not on the entire past. Markov chains have many applications as statistical models of real-world processes and can be applied to tennis.

Each score in a game of tennis is a state of the system as represented in table 1. Note that (40-30) and ad-in, (30,30) and deuce, and (30,40) and ad-out are equivalent states.

State	Score
0	(0,0)
1	(15,0)
2	(0,15)
3	(30,0)
4	(15,15)
5	(0,30)
6	(40,0)
7	(30,15)
8	(15,30)
9	(0,40)
10	(40,15)
11	(15,40)
12	game server
13	game receiver
14	(40,30) or ad-in
15	(30,30) or deuce
16	(30,40) or ad-out

Table 1: Scores in a game of tennis represented as states of a system

Let p represent the probability of the server winning a point and q=1-p.

The transition matrix given by table 2 represents the probability $P_{i,j}$ that the process will, when in state i, next make a transition into state j. For example the probability $P_{0,1}$ of going from state 0 (score (0,0)) to state 1 (score (15,0)) is p. Similarly, the probability $P_{0,2}$ of going from state 0 (score (0,0)) to state 2 (score (0,15)) is q. Note that $\sum_i P_{i,j}$ =1.

We have already defined the one-step transition probabilities $P_{i,j}$. We now define the n-step transition probabilities $P^n_{i,j}$ to be the probability that a process in state i will be in state j after n additional transitions. The Chapman-Kolomogorov equations provide a method for computing these n-step transition probabilities and is shown that the n-step transition matrix may be obtained by multiplying the matrix by itself n times. Table 3 represents the transition matrix $P^2_{i,j}$. For example $P^2_{0,3}$ is the probability of being at score line (30,0) from score line (0,0) after two points have been played. From table 3 this is given as p^2 as expected.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0	p	q	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	p	q	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	p	q	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	p	q	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	p	q	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	p	q	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	q	0	p	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	p	0	0	0	0	q	0
8	0	0	0	0	0	0	0	0	0	0	0	q	0	0	0	p	0
9	0	0	0	0	0	0	0	0	0	0	0	p	0	q	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	p	0	q	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	q	0	0	p
12	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	p	0	0	q	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	p	0	q
16	0	0	0	0	0	0	0	0	0	0	0	0	0	q	0	p	0

Table 2: Transition matrix for a game of tennis

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0	0	0	p^2	$2pq$	q^2	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	p^2	$2pq$	q^2	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	p^2	$2pq$	q^2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	$2pq$	0	p^2	0	0	q^2	0
4	0	0	0	0	0	0	0	0	0	0	p^2	q^2	0	0	0	$2pq$	0
5	0	0	0	0	0	0	0	0	0	0	0	$2pq$	0	q^2	0	p^2	0
6	0	0	0	0	0	0	0	0	0	0	0	0	$pq+p$	0	q^2	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	p^2	0	$2pq$	0	q^2
8	0	0	0	0	0	0	0	0	0	0	0	0	0	q^2	p^2	0	$2pq$
9	0	0	0	0	0	0	0	0	0	0	0	0	0	$pq+p$	0	0	p^2
10	0	0	0	0	0	0	0	0	0	0	0	0	$pq+p$	0	0	q^2	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	$pq+p$	0	p^2	0
12	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	p	0	pq	0	q^2
15	0	0	0	0	0	0	0	0	0	0	0	0	p^2	q^2	0	$2pq$	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	q	p^2	0	pq

Table 3: Transition matrix $P^2_{i,j}$ for a game of tennis

Note that $\sum_j P^n_{i,j} = 1$

Using a four-step transition matrix:
Probability server winning to 0 = $P^4_{(0,12)} = p^4$
Probability receiver winning to 0 = $P^4_{(0,13)} = q^4$

Using a five-step transition matrix:
Probability server winning to 15 = $P^5_{(0,12)} - P^4_{(0,12)} = (4p^4q+p^4)-p^4=4p^4q$
Probability receiver winning to 15 = $P^5_{(0,13)} - P^4_{(0,13)} = (4q^4p+q^4)-q^4=4q^4p$

Using a six-step transition matrix:
Probability server winning to 30 = $P^6_{(0,12)}-(P^5_{(0,12)}- P^4_{(0,12)})- P^4_{(0,12)}= P^6_{(0,12)}-P^5_{(0,12)}=(10p^4q^4+4p^4q+p^4)-(4p^4q+p^4)=10p^4q^2$
Probability receiver winning to 30= $P^6_{(0,13)}-(P^5_{(0,13)}- P^4_{(0,13)})- P^4_{(0,13)}= P^6_{(0,13)}-P^5_{(0,13)}=(10q^4p^4+4q^4p+q^4)-(4q^4p+q^4)=10q^4p^2$
Probability reaching deuce = $P^6_{(0,15)}=20p^3q^3$

Hence the probability of the server winning the game prior to deuce is given by $p^4+4p^4q+10p^4q^2 = p^4(1+4q+10q^2)$

Note the probability of winning from deuce using Markov Chain theory is outlined in Kemeny and Snell[1]

Reference

[1] Kemeny, J. and Snell, J. Finite Markov Chains. Van Nostrand, Princeton, NJ. 1976, p163