Winning a game of tennis: using the Binomial theorem

In elementary algebra, the Binomial theorem describes the algebraic expansion of powers of a binomial. According to the theorem, it is possible to expand the powers of a binomial $(p+q)^n$, for $n > 0$: integer.

The Binomial coefficients appear as the entries of Pascal's triangle. The diagram below gives the first 6 rows of Pascal's triangle where the rows are conventionally enumerated starting with row $n = 0$ at the top. The triangle is constructed such that the only element in row 0 is a 1 and the elements in each subsequent row are obtained by adding the number directly above and to the left with the number directly above and to the right. If either the number to the right or left is not present, substitute a zero in its place. For example the coefficients of $(p+q)^6$ are given by the 6th row.

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

More formally, how many ways can *n* objects be chosen *r* at a time is equal to the Binomial coefficient $_nC_r = n!/(r!(n-r)!)$, where $k! = k\times(k-1) \times (k-2) \times \ldots\times 3\times 2$ for any integer $k \geq 2$, and $1! = 0! = 1$.
Hence the Binomial theorem is given by:

$$(p+q)^n = \sum_{r=0}^{n} nCrp^r q^{n-r}$$

For example: $(p+q)^6$
$= {}_6C_0\, p^0q^6 + {}_6C_1\, p^1q^5 + {}_6C_2\, p^2q^4 + {}_6C_3\, p^3q^3 + {}_6C_4\, p^4q^2 + {}_6C_5\, p^5q^1 + {}_6C_6\, p^6q^0$
$= q^6 + 6pq^5 + 15p^2q^4 + 20p^3q^3 + 15p^4q^2 + 6p^5q + p^6$

Notice how the coefficients agree with the 6th row in Pascal's triangle.

A Binomial experiment possesses the following properties:
1. The experiment consists of *n* identical trials
2. Each trial results in one of two outcomes; either a success *S* or a failure *F*
3. The probability of success on a single trial is equal to *p* and remains the same from trial to trial. The probability of a failure is equal to $q = 1-p$.
4. The trials are independent.

5. The random variable of interest is *R*, the number of successes observed during *n* trials.

The Binomial theorem can be applied to a tennis game since there are two outcomes; either the server wins a point or receiver wins a point, the probability of success on a single trial *p* of the server winning a point is constant from trial to trial, the probability of a failure is equal to $q = 1\text{-}p$, and the game consists of *n* identical and independent trials.

Suppose four points are played - then either there are zero successes by the receiver winning all four points and the game, one success by the server winning one point and the receiver winning three points for a score line of (15,40), two successes by the server and receiver winning two points each for a score line of (30,30), three successes by the server winning three points and the receiver winning one point for a score line of (40,15), or four successes by the server winning all four points and the game. Using the Binomial theorem:

$$(p+q)^4 = \sum_{r=0}^{4} 4\text{C}r\, p^r q^{4-r}$$

$= {}_4C_0\, p^0q^4 + {}_4C_1\, p^1q^3 + {}_4C_2\, p^2q^2 + {}_4C_3\, p^3q^1 + {}_4C_4\, p^4q^0$

$= q^4 + 4pq^3 + 6p^2q^2 + 4p^3q + p^4$

Table 1 represents the probability outcomes of a game of tennis from the Binomial theorem with $n = 4$. It shows the probability that the server wins the game to 0 is p^4. Alternatively, if the server wins to 0, then the game must last exactly four points and the server must win all four points. This is the situation of the number of ways 4 objects be chosen 4 at a time and occurs in ${}_4C_4 = 1$ way. Note that ${}_4C_4 = 1$ is given by the last entry in the 4th row in Pascal's triangle. Hence the server wins the game to 0 with probability ${}_4C_4p^4q^{4\text{-}4} = p^4$.

Suppose five points are played - then either there is one success by the server winning one point and the receiver winning four points and the game (which must include the 5^{th} point), two successes by the server winning two points and the receiver winning three points for a score line of (30,40), three successes by the server winning three points and the receiver winning two points for a score line of (40,30), or four successes by the server winning four points and the game (which must include the 5^{th} point) and the receiver winning one point. Note that zero or five successes cannot occur since the server or receiver has already won the game after four consecutive points won. Using the Binomial theorem:

$({}_5C_1 - {}_4C_0)p^1q^4 + {}_5C_2p^2q^3 + {}_5C_3p^3q^2 + ({}_5C_4 - {}_4C_4)p^4q^1$
$= 4pq^4 + 10p^2q^3 + 10p^3q^2 + 4p^4q$

Successes	Score line	Probability
0	Game receiver	q^4
1	(15,40)	$4pq^3$
2	(30,30)	$6p^2q^2$
3	(40,15)	$4p^3q$
4	Game Server	p^4

Table 1: Probability outcomes of a game of tennis from the Binomial theorem with *n*=4

Therefore the probability that the server wins the game to 15 is $4p^4q$. Alternatively, if the server wins to 15, then the game must last exactly five points and the server must win four points (which must include the 5th point). This is the situation of the number of ways 5 objects be chosen 4 at a time minus the number of ways 4 objects be chosen 4 at a time (server winning to 0), and occurs in ${}_5C_4$-${}_4C_4$=5-1=4 ways. Hence the server wins the game to 15 with probability $({}_5C_1 - {}_4C_4)p^4q^{5-4} = 4p^4q$.

If the server wins to 30, then the game must last exactly six points and the server must win 4 points (which must include the 6th point). This is the situation of the number of ways 6 objects be chosen 4 at a time minus the number of ways 5 objects be chosen 4 at a time (server winning to 15) minus the number of ways 4 objects be chosen 4 at a time (server winning to 0) and this occurs in ${}_6C_4$-${}_5C_4$-${}_4C_4$=15-4-1=10 ways. Hence the server wins the game to 30 with probability $({}_6C_4\text{-}{}_5C_4\text{-}{}_4C_4)p^4q^{6-4} = 10p^4q^2$.

For the score to reach deuce, the server must win three of the first six points played and this occurs in ${}_6C_3$=20 ways. Hence the probability of reaching deuce is ${}_6C_3p^3q^{6-3} = 20p^3q^3$.

To win the game from deuce the server needs to win the next two points. This occurs with probability p^2. If the server wins the next point from deuce followed by the receiver winning a point, or the receiver wins the next point from deuce followed by the server winning a point, then the score returns to deuce. This occurs with probability $2pq$. If d is the probability that the server wins the game when the score is at deuce, then $d = p^2+2pqd$.

Solving for d gives $d =p^2/(1\text{-}2pq)$

Using the fact that $p^2 + q^2 = (p + q)^2 - 2pq = 1\text{-}2pq$, we can also write $d =p^2/(p^2+q^2)$

Therefore the probability of the server winning the game from the outset (0,0) is given by:
$p^4(1 + 4q + 10q^2) + 20p^3q^3\, p^2/(p^2+q^2)= p^4(1 + 4q +10q^2/(p^2+q^2))$

The simplification arises since (30,30) and deuce are logically equivalent score lines.

Table 2 represents the probabilities of the server winning the game for different values of p.

Probability of server winning a point	Probability of server winning a game
0.50	0.500
0.55	0.623
0.60	0.736
0.65	0.830
0.70	0.901
0.75	0.949
0.80	0.978

Table 2: Probabilities of server winning a game for different values of p

Notice from table 2 how difficult it is for the receiver to break server when the p values are greater than 0.75. This typically occurs in men's tennis and demonstrates why long final advantage sets occur in grand slam matches.