

OPTIMAL USE OF TENNIS RESOURCES

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Abstract

This paper demonstrates how tennis players can optimize their chance of winning a match by using strategies to utilize their energy resources. This can be achieved by either increasing effort on certain points, games and sets in a match, or by increasing and decreasing effort about an overall mean. The results show that increasing effort on any point in a game before deuce is reached has the same effect on the player's chances of winning the game. By increasing effort on the important points and decreasing effort on the unimportant points in a game, players can increase their chances of winning a game. For the better player, this gain is a result of the variability about the mean and also the importance of points. The results obtained in tennis are used to investigate problems related to warfare.

1. Introduction

The scoring structure of tennis is designed in such a way, that to win, a player must reach a winning score rather than be ahead at a predetermined time. For a 5 set match, the player that first reaches 3 sets wins the match. To win a classical or advantage set, a player must win at least 6 games and be ahead by at least 2 games. And finally to win a game, a player must win at least 4 points and be ahead by at least 2 points. As a consequence of this scoring system, it is possible for a player to win the match by winning well under half of the points played (Ferris [2]). This raises questions about whether some points, games and sets are more important than others, and whether players should distribute their energy resources accordingly?

Tennis commentators often state that the best players can raise their level when faced with break points, e.g 30-40, 15-40. A counter argument is that if these players raised their level earlier in the game, then they could avoid being down break points altogether. Morris [3], O'Donoghue [4] and Pollard [5] show that expending additional physical and mental effort on the important points in a game whilst relaxing on the unimportant points increases the chances of winning a game. In particular Morris [3] states "*If he increased p from 0.60 to 0.61 on half his service points, and decreased from 0.60 to 0.59 on the unimportant half, he would increase his winning percentage by 0.0075 from 0.7357 to 0.7432*". We demonstrate that this increase in the chances of winning the game is produced as a result of the variability about the mean p , as well as the importance of points to winning a game.

Sections 2 and 3 establish some important results. This is achieved by calculating the conditional probabilities of players winning a best of 3 set match for a constant probability of winning a set. Then calculations are produced for the conditional probabilities of players winning the match by increasing probability, or increasing and decreasing probability on certain sets within the match. Definitions and properties of importance, time-importance and weighted-importance are produced in section 3. Section 4 uses the established results to solve problems on utilizing energy resources. Sets within a match are considered first. Optimal strategies are determined for players increasing, or increasing and decreasing effort on certain sets within a match. Similar problems are solved for points in a game. In section 5, the results obtained in tennis are used to investigate problems related to warfare. The paper concludes in section 6 with a summary and a general discussion.

2. Probabilities on winning a match

A best of 3 set match is a contest where the first player to win 2 sets wins the match. Analyzing this system is non trivial despite its relatively simple structure, because it is not certain that the third set will be played.

Consider the situation where a player has a constant probability p of winning a set. What is the probability of this player winning a 3 set match? The player can win the match by either winning in straight sets with probability p^2 , losing the first set and winning the last two sets with probability $(1-p)p^2$ or winning the first and last set and losing the second set with probability $p(1-p)p$. Summing these, the chance of the player to win the match is given by $p^2(3-2p)$.

An alternative method, described in Barnett and Clarke [1], uses recursion to calculate the probability of a player winning the match from any position. This method allows more flexibility and easier calculation when the probability of winning a set is not constant.

If $P(e,f)$ is the probability that the referred player wins the match when the match score in sets is (e,f) (e =referred player's score, f =opponent's score), the recurrence formula becomes:

$$P(e,f) = pP(e+1,f) + (1-p)P(e,f+1)$$

The boundary values are $P(2,0)=P(2,1)=1$, $P(0,2)=P(1,2)=0$.

Table 1 represents the conditional probabilities of a player winning a match. The match score at $(0,0)$ agrees with our former calculation of $p^2(3-2p)$.

Table 1: The conditional probabilities $P(e,f)$ of players winning a match.

		opponent score in sets		
		0	1	2
player score in sets	0	$p^2(3-2p)$	p^2	0
	1	$p(2-p)$	p	0
	2	1	1	

Similar calculations are used to calculate the probabilities of reaching score lines within a match. If $N(e,f/k,l)$ is the probability of reaching a match score (e,f) in a match from match score (k,l) , the recurrence formulas become:

$$\begin{aligned}
 N(e,f/k,l) &= pN(e-1,f/k,l) && \text{for } e = 2, \text{ or } f = 0 \\
 N(e,f/k,l) &= (1-p)N(e,f-1/k,l) && \text{for } e = 0, \text{ or } f = 2 \\
 N(e,f/k,l) &= pN(e-1,f/k,l) + (1-p)N(e,f-1/k,l) && \text{for } e = 1, f = 1
 \end{aligned}$$

The boundary values are $N(e,f/k,l) = 1$ if $e = k$ and $f = l$.

Table 2 lists the probabilities of reaching various score lines in a match given $k=0, l=0$. The probability of playing the first and second set is always one, but the probability of playing the third set is $2p(1-p) < 1$.

Table 2: The probabilities $N(e,f/0,0)$ of reaching various score lines in a match.

		opponent score in sets		
		0	1	2
player score in sets	0	1	$1-p$	$(1-p)^2$
	1	p	$2p(1-p)$	$2p(1-p)^2$
	2	p^2	$2p^2(1-p)$	

Now suppose a player increases his effort for one set at a match score (e,f) , so as to change his probability of winning this set from p to $p+\epsilon$, where $p+\epsilon < 1$. This is equivalent to the opponent decreasing his effort at a match score (e,f) so as to change his probability of winning this set from $1-p$ to $1-p-\epsilon$, since an increase of the probability of winning to one player is a decrease to the other player. If the increase in effort is applied at $(0,0)$, the chance for the player to win the match becomes $(p+\epsilon)p(2-p) + (1-p-\epsilon)p^2 = p^2(3-2p) + \epsilon 2p(1-p)$. The same result is obtained if an increase in effort is applied at $(1,1)$. Similarly the chances of a player to win the match when an increase in effort is applied at one of $(1,0)$ or $(0,1)$ is $p^2(3-2p) + \epsilon p(1-p)$. Conditional on the match score reaching $(1,0)$, the chance for a player to win the match when an increase in effort is applied at $(1,0)$ or $(1,1)$ is $p(2-p) + \epsilon(1-p)$; and conditional on the match score reaching $(0,1)$, the chance for a player to

win the match when an increase in effort is applied at (0,1) or (1,1) is $p^2 + \varepsilon p$. Table 3 gives the increase in probability when effort is applied throughout the match.

Table 3: The increase in probability when effort is applied throughout the match.

Current match score	Match score at which an increase is applied	Increase in probability of winning match
(0,0)	(0,0)	$\varepsilon 2p(1-p)$
	(1,0)	$\varepsilon p(1-p)$
	(0,1)	$\varepsilon p(1-p)$
	(1,1)	$\varepsilon 2p(1-p)$
(1,0)	(1,0)	$\varepsilon(1-p)$
	(1,1)	$\varepsilon(1-p)$
(0,1)	(0,1)	εp
	(1,1)	εp
(1,1)	(1,1)	ε

The first set played begins with the match score at (0,0). The third set is played only if the match score reaches (1,1). The second set played occurs with the match score at either (1,0) or (0,1). The chance of a player winning the match when one increase in effort is applied on the first, second or third set played is equal to $p^2(3-2p) + \varepsilon 2p(1-p)$.

Now suppose a player adopts a strategy of increasing his effort on the first, second or third set played by ε , and decreases p on the first, second or third set played (but a different set played from that of the increase) by ε , where $0 < p+\varepsilon < 1$. Calculations show the chance of the player winning the match for this situation is equal to $p^2(3-2p) + \varepsilon^2(2p-1)$.

3. Importance, Time-Importance and Weighted-Importance

3.1 Notation for points, games, sets and matches

Tennis is a game where the structure of the scoring system is hierarchical. It becomes convenient in the following sections to repeat the use of a symbol at various levels, and to distinguish between the levels by means of accents. For example, let p be the probability of a player winning a point, p' be the probability of a player winning a game, p'' be the probability of a player winning a set, and p''' be the probability of a player winning a match.

Subscripts are used where it is necessary to distinguish between a player and his opponent, e.g. p_A and p_B are the probabilities of players A and B winning a point on serve respectively.

3.2 Importance and Time-Importance

Morris [3] defines the importance of a point to winning the game $I(a,b)$ as the probability a player wins the game given that he wins the point, minus the probability that he wins the game given he loses the point. If $P(a,b)$ is the probability a player wins the game from game score (a,b) , this can be represented by $I(a,b) = P(a+1,b) - P(a,b+1)$. It follows that the importance of a set to winning the match is represented by: $I''(e,f) = P''(e+1,f) - P''(e,f+1)$. These formulas apply to both players, since every point in a game or set in a match is equally important for both players (Morris [3]). Table 4 lists the importance of a set to winning the match and shows clearly the set (1,1) has the highest importance to winning a match. If $p'' > 1/2$, then (0,1) is more important than (1,0).

Table 4: The importance of a set to winning the match.

		opponent score in sets	
		0	1
player score in sets	0	$2 p''(1-p'')$	p''
	1	$1-p''$	1

Morris [3] defines time-importance using the following formula when $g=0, h=0$:

$$T(a,b/g,h) = I(a,b)E(a,b/g,h)$$

where $T(a,b/g,h)$ is the time-importance of point (a,b) in a game from game score (g,h) for a player, and $E(a,b/g,h)$ is the expected number of times the point (a,b) is played in a game from game score (g,h) . With this definition, Morris [3] considers advantage server to be the same score as $(a=3,b=2)$, since these points are logically equivalent. Similarly he considers advantage receiver to be the same score as $(a=2,b=3)$.

3.3 Weighted-Importance

We define weighted-importance using the following formula:

$$W(a,b/g,h) = I(a,b)N(a,b/g,h)$$

where $W(a,b/g,h)$ is the weighted-importance of point (a,b) to winning the game for a player from game score (g,h) , $N(a,b/g,h)$ is the probability of reaching point (a,b) in a game for a player from game score (g,h) . It follows that $W''(e,f/k,l) = I''(e,f)N''(e,f/k,l)$ is the weighted-importance of set (e,f) to winning the match from match score (k,l) . It also follows that $W''(e,f/k,l) = T''(e,f/k,l)$ for all e,f,k,l but $W(a,b/g,h) \neq T(a,b/g,h)$ for all a,b,g,h .

Table 5 represents the weighted-importance of sets in a match from match score $(k=0,l=0)$ and notice that $W''(0,0|0,0)$, $W''(1,1|0,0)$ and $W''(1,0|0,0)+W''(0,1|0,0)$ all equal $2p''(1-p'')$.

Table 5: The weighted-importance of sets in a match from $(0,0)$.

		opponent score in sets	
		0	1
player score in sets	0	$2p''(1-p'')$	$p''(1-p'')$
	1	$p''(1-p'')$	$2p''(1-p'')$

Morris [3] states the following formula for time-importance when $g=0, h=0$:

$$\Sigma_{(a,b)} T(a,b/g,h) = \frac{dP(g,h)}{dp}$$

We introduce the corresponding formula on weighted-importance:

$$\Sigma_{(e,f)} W''(e,f/k,l) = \frac{dP''(k,l)}{dp''}$$

Morris [3] states the following property about time-importance. Suppose a server, who ordinarily has probability p of winning a point on his serve, decides that he will try harder every time the point (a,b) occurs. If by doing so he is able to raise his probability of winning from p to $p+\varepsilon$, ($\varepsilon > 0$ but small) for that point alone, then he raises his probability of winning the game from $P(g=0,h=0)$ to $P(g=0,h=0) + \varepsilon T(a,b|g=0,h=0)$.

Likewise it can be shown that for weighted-importance of sets to winning a match, has the following property:

Property 1: Suppose a player, who ordinarily has probability p'' of winning a set, decides that he will try harder every time the set (e,f) occurs. If by doing so he is able to raise his probability of winning from p'' to $p'' + \varepsilon$, ($p'' + \varepsilon < 1$) for that set alone, then he raises his probability of winning the match from $P''(k,l)$ to $P''(k,l) + \varepsilon W''(e,f/k,l)$.

A similar property can be obtained for points in a game (Property 2).
 Because $N''(e,f|k=e,l=f)=1$, Property 3 arises as a special case of Property 1.

Property 3: Suppose a player, who ordinarily has probability p'' of winning a set, decides that he will try harder every time the set (e,f) occurs. If by doing so he is able to raise his probability of winning from p'' to $p'' + \varepsilon$, ($p'' + \varepsilon < 1$) for that set alone, then he raises his probability of winning the match from $P''(e,f)$ to $P''(e,f) + \varepsilon I''(e,f)$.

Morris [3] states the importance of winning a point to winning the match $I''(a,b:c,d:e,f)$ can be obtained from the importance of a point to winning the game $I(a,b)$, the importance of a game to winning the set $I(c,d)$ and the importance of a set to winning the match $I''(e,f)$ as follows:

$$I''(a,b:c,d:e,f) = I(a,b)I(c,d)I''(e,f) \dots\dots\dots(1)$$

4. Interpretation of results

4.1 Sets in a match

Suppose a player can apply an increased effort in a match on any set played so as to increase p'' to $p'' + \varepsilon$, $p'' + \varepsilon < 1$. On which set, should the player apply the increase to optimize their chances of winning the match?

From Table 3, applying an increased effort at (0,0) or (1,1) results in an increased chance of $\varepsilon 2 p'' (1 - p'')$. Applying an increase in effort at (1,0) or (0,1) results in an increased chance of only $\varepsilon p'' (1 - p'')$. However (1,0) and (0,1) constitute the second set played and the sum of these increases is equivalent to $\varepsilon 2 p'' (1 - p'')$. Also conditional on the match score reaching (1,0), increasing effort on (1,0) or (1,1) results in an increase of $\varepsilon(1 - p'')$ and conditional on the match score reaching (0,1), increasing effort on (0,1) or (1,1) results in an increase of $\varepsilon p''$. Therefore an increase in effort could be applied at either the first, second or third set played to optimize the chances for the player to win the match. The same conclusion can be obtained by looking at the weighted-importance of sets in a match as a result of Property 1.

Suppose a player can apply M increases of effort in a match, $0 < M \leq 3$, on any set/s played by increasing p'' to $p'' + \varepsilon$, $p'' + \varepsilon < 1$. On which set/s, should the player apply increases to optimize their chances of winning the match?

Since it is optimal to apply an increase on any set played, an optimal strategy is to apply the M increases on every set played throughout the course of the match until there are no increases remaining.

The chance of a player winning a match based on a constant probability is given by $p''^2(3 - 2 p'')$. When an increase and a corresponding decrease in effort is applied in any order to the first, second or third set played, there is an additional term in the chances of winning the match of $\varepsilon^2(2 p'' - 1)$. When $p'' = 1/2$, $2 p'' - 1 = 0$, and there is no change in the chances for either player to win the match. When $p'' > 1/2$, the chance for the player to win the match increases by $\varepsilon^2(2 p'' - 1)$ and therefore the opponent's chances to win the match decrease by $\varepsilon^2(2 p'' - 1)$. This implies that it is an advantage for the better player to vary his effort whilst maintaining his mean probability of winning a set. It follows by symmetry that the weaker player is disadvantaged by varying his effort.

The weighted-importance of the first set is $2 p''(1 - p'')$, which is the same as the weighted-importance of the second set. Since these sets are always played in a 3 set match, the chances of a player winning the match when an increased effort is applied on the first set and a corresponding decreased effort on the second set, is the same as when a decrease in effort is applied on the first set and an increase in effort on the second set. In this situation, the increase or decrease in probability of winning the match for a player is caused by the variation about the mean probability of winning a set. However this is not the case for the third set played which has the highest importance in the match. This set is only played a

proportion of the time, and the better player could further increase his chance of winning the match by increasing their effort on the third set played and a proportion of the time on the second set played.

For example, if a player has a probability of winning a set given by $p'' = 0.6$, then the probability of this player winning the match is 0.648. If a decrease in probability by $\varepsilon = 0.1$ occurs on the first set and an increase in probability by $\varepsilon = 0.1$ occurs on the third set, then the probability of this player winning the match becomes 0.650. However, since the third set is only played a proportion of the time, additional increase in effort can also be applied on the second set with probability z , where z is found by solving the equation: $0.5(0.7z+0.6(1-z))+0.5(1-(0.7z+0.6(1-z)))+z=1$, i.e. $z=0.5$, in which case the probability of this player winning the match now becomes 0.675. Out of the $0.675-0.648 = 0.027$ increase in probability of winning the match for this player, $(0.675-0.650)/0.027 = 92.59\%$ is contributed by the fact of the third set being more important than the other sets. Similar calculations show that the opponent with a probability of 0.4 of winning a set also gains an advantage by decreasing effort on the first set and increasing effort on the third set and a proportion of the time on the second set. But the increase gained of 0.025 is less than what their opponent achieves since $p'' > 1/2$.

4.2 Points in a game

Suppose a player can apply an increased effort in a game on any one point played by increasing p to $p+\varepsilon$, $p+\varepsilon < 1$. On which point, should the player apply the increase to optimize their chances of winning the game?

Property 2 about weighted-importance of points in a game is used to solve this problem. Recurrence formulas for a game are entered in a spreadsheet and $p = 0.61$ is used to represent the men's average percentage of points won on serve. Table 6 represents the weighted-importance of points in a game for $p = 0.61$. The sum of the group of points for the n^{th} point played (represented by the diagonals) are equal to 0.257 for $n \leq 6$ and 0.122 for $n = 7, 8$. It can be verified that $W(a,b/g,h)$ for all g,h give similar results. Therefore a player can increase their effort on any point in the game, providing this increase is applied before deuce is reached, and they have optimized the usage of their one available increase.

Table 6: The weighted-importance of points in a game from (0,0) and $p=0.61$

		opponent score in points				Ad
		0	15	30	40	
player score in points	0	0.257	0.134	0.057	0.016	
	15	0.123	0.154	0.124	0.063	
	30	0.046	0.107	0.154	0.157	
	40	0.010	0.040	0.100	0.122	0.075
	Ad				0.048	

Suppose a player with $p=0.61$ can apply M increases in a game, on any point/s played by increasing p to $p+\varepsilon$, $p+\varepsilon < 1$. On which point/s, should the player apply an increase in effort to optimize their chances of winning the game?

It is optimal to apply an increase on any point in a game, providing the increase is applied before deuce is reached. It can also be shown that the sum of the weighted-importance of points for the $n = 7+2m$ points played is equal to the sum of the weighted-importance of points for the $n = 8+2m$ points played and less than the sum of the weighted-importance of points for the $n = 9+2m$ points played for all $m \geq 0$. Therefore an optimal strategy for a player is to apply an increase in effort on every point in the game until either the game is finished or there are no increases remaining.

Since the sum of the weighted-importance of points for $n \leq 6$ are all equal, then the chances of a player winning the game are the same irrespective of which points played an increase and decrease in effort is applied, providing $n \leq 6$. However, similar to sets in a match, a player can gain a significant advantage by increasing effort at $n = 5$ or 6 due to the fact that $n = 5$ or 6 only occurs a proportion of the time. It can be shown that for a player on serve with $p=0.61$, 30-40 is the most important point in a game, followed by 30-30 and deuce. The least important point is 40-0. Therefore a player can gain a

significant advantage by increasing effort on the important points in a game and decreasing effort on the unimportant points.

5. Applications to warfare

This paper is a result of a problem organized by the Defence Science of Technology Organization at the 2003 Mathematics in Industry Study Group (MISG). The following analogue between tennis and warfare is proposed by representing a tennis match as a war conflict, where the levels of hierarchies in a tennis match consisting of points, games, sets and match now become skirmishes, battles, campaigns and war. However in warfare, there are costs associated in applying an increase on points, games or sets. This could involve the cost of firing an extra missile.

In the model it is assumed there are a large number of increases in effort available for use, and if the allocated M increases run out, the supply can always be replenished. There is a reward r for winning the overall war and a cost c for applying an increase at a particular skirmish. Ultimately the hope is to win the war by applying M increases, to maximize $r-Mc$. There might be a good chance of winning the war by applying an increase on every skirmish, but overall the war might be a financial loss because of the high costs associated with the large number of increases. Clearly there is a trade off between the value of winning the war and the number of increases in effort that are applied.

In order to see what can be learnt from this model of warfare, it is convenient to return to the tennis contest. For each point of the match, a decision must be made on whether it is worth applying an increase, to maximize the expected payout of the match.

Firstly, consider sets within a match, where g = the cost of applying an increased effort to a set in the match.

Let $EX(e,f)=[pP(e+1,f)+(1-p)P(e,f+1)]r$ and $EX_I(e,f)=[(p+\varepsilon)P(e+1,f)+(1-p-\varepsilon)P(e,f+1)]r-g$, where $EX(e,f)$ = expected payout at (e,f) in a match with no increase and $EX_I(e,f)$ = expected payout at (e,f) in a match with an increase. If $EX_I(e,f) - EX(e,f) > 0$ then an increase should be applied at (e,f) . $EX_I(e,f) - EX(e,f)$ simplifies to $\varepsilon[P(e+1,f) - P(e,f+1)]r-g$, which is equivalent to $\varepsilon I(e,f)r-g$. This implies that an increase should be applied at (e,f) if $\varepsilon I(e,f)r-g > 0$, or equivalently if

$$I(e,f) > \frac{g}{r\varepsilon}$$

Similar definitions can be obtained for applying an increase for points in a game and games in a set and an increase should be applied on points in the match for which

$$I(a,b:c,d:e,f) > \frac{c}{r\varepsilon}$$

6. Conclusions

This paper has demonstrated that strategies do exist in tennis as a result of the scoring structure. It has been shown that a player can increase their effort on any point in a game before deuce, and they have optimized the usage of this one available increase. It has also been shown that an increased chance of a player winning a game by varying effort on the first, second, third or fourth points played, for $p > 1/2$, is due to the variation about the mean p . However, since the fifth or sixth points played only occur a proportion of the time, the better player can obtain an even greater advantage by increasing effort on the most important points and decreasing effort on the least important points in a game. A model is formulated from the results obtained in a tennis match, to show how extra resources can be utilized in warfare.

There are many other problems that arise from the problems considered in this paper. Examples include the effect on the chances of players winning the match (if any) of momentum or morale, depleting available capability through the effort to win the point, or other psychological effects. The generality of the formulae

and properties obtained throughout this paper, make it feasible to analyze other scoring structures that may be more applicable to warfare.

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