Modelling outcomes in badminton

Scoring system

Match	Best-of-3 games
Game	First to 21 points At 20-all, 2 point lead At 29-all, next point
Server in game	Winner of point
Serving first each game	Winner of game

Chances of winning a game

Let p_A represent the probability of player A winning a point on serve
Let p_B represent the probability of player B winning a point on serve

Let $q_A=1-p_A$ and $q_B=1-p_B$

Let $P_A^{pg}(a,b)$ and $P_B^{pg}(a,b)$ represent the probabilities of player A winning a game at point score (a,b) for player A and player B serving respectively

Recurrence formulas:
$P_A^{pg}(a,b) = p_A P_A^{pg}(a+1,b)+q_A P_B^{pg}(a,b+1)$
$P_B^{pg}(a,b) = p_B P_B^{pg}(a,b+1) +q_B P_A^{pg}(a+1,b)$

Boundary values:
$P_A^{pg}(a,b) = P_B^{pg}(a,b) = 1$ if $a=21$ and $b \leq 19$ or (a,b) = (22,20), (23,21), (24,22), (25,23), (26,24), (27,25), (28,26), (29,27), (30,28), (30,29)
$P_A^{pg}(a,b) = P_B^{pg}(a,b) = 0$, if $b=21$ and $a \leq 19$ or (a,b) = (20,22), (21,23), (22,24), (23,25), (24,26), (25,27), (26,28), (27,29), (28,30), (29,30)
$P_A^{pg}(29,29) = p_A$
$P_B^{pg}(29,29) = q_B$

Chances of winning a match

Let $P_A^{pg}(0,0) = p_A^g$
Let $P_B^{pg}(0,0)=q_B^g$

Let $q_A^g=1- p_A^g$ and $p_B^g=1-q_B^g$

Let $P_A^{gm}(c,d)$ and $P_B^{gm}(c,d)$ represent the probabilities of player A winning a match at game score (c,d) for player A and player B serving respectively

Recurrence formulas:
$P_A^{gm}(c,d)= p_A^g P_A^{gm}(c+1,d)+q_A^g P_B^{gm}(c,d+1)$
$P_B^{gm}(c,d)=q_B^g P_A^{gm}(c+1,d)+p_B^g P_B^{gm}(c,d+1)$

Boundary values:
$P_A^{gm}(2,0)=P_B^{gm}(2,0)=1$
$P_A^{gm}(0,2)=P_B^{gm}(0,2)=0$
$P_A^{gm}(1,1)= p_A^g$
$P_B^{gm}(1,1)= q_B^g$

Let Let $P_A^{pm}(a,b:c,d)$ and $P_B^{pm}(a,b:c,d)$ represent the probabilities of player A winning a match at point and game score (a,b:c,d) for player A and player B serving respectively

$P_A^{pm}(a,b:c,d)= P_A^{pg}(a,b)P_A^{gm}(c+1,d)+(1- P_A^{pg}(a,b))P_B^{gm}(c,d+1)$
$P_B^{pm}(a,b:c,d)= P_B^{pg}(a,b) P_A^{gm}(c+1,d)+(1-P_B^{pg}(a,b)) P_B^{gm}(c,d+1)$

p_A	p_B	$P_A^{pg}(0,0)$	$P_B^{pg}(0,0)$	$P_A^{gm}(0,0)$	$P_B^{gm}(0,0)$
0.46	0.46	0.495	0.505	0.497	0.503
0.48	0.46	0.549	0.556	0.577	0.581
0.50	0.46	0.602	0.607	0.653	0.656
0.52	0.46	0.653	0.655	0.723	0.725
0.48	0.48	0.497	0.503	0.499	0.501
0.50	0.48	0.551	0.554	0.578	0.579
0.52	0.48	0.604	0.604	0.654	0.654
0.50	0.50	0.500	0.500	0.500	0.500
0.52	0.50	0.554	0.551	0.579	0.578
0.52	0.52	0.503	0.497	0.501	0.499

Table: Probabilities of player A winning a game and match for different values of p_A and p_B, and for both player A and player B serving first

Mean number of points remaining in the game

Let $\mu(Y_A^{pg}(a,b))$ and $\mu(Y_B^{pg}(a,b))$ represent the mean number of points remaining in a game at point score (a,b) for player A and player B serving respectively

Recurrence formulas:
$\mu(Y_A^{pg}(a,b))=1+p_A\ \mu(Y_A^{pg}(a+1,b))+q_A\ \mu(Y_B^{pg}(a,b+1))$
$\mu(Y_B^{pg}(a,b))=1+p_B\ \mu(Y_B^{pg}(a,b+1)) +q_B\ \mu(Y_A^{pg}(a+1,b))$

Boundary values:
$\mu(Y_A^{pg}(a,b))=\mu(Y_B^{pg}(a,b))=0$, if a=21 and b ≤19 or b=21 and a ≤19 or (a,b) = (22,20), (23,21), (24,22), (25,23), (26,24), (27,25), (28,26), (29,27), (30,28), (20,22), (21,23), (22,24), (23,25), (24,26), (25,27), (26,28), (27,29), (28,30)
$\mu(Y_A^{pg}(29,29))=\mu(Y_B^{pg}(29,29))=1$

Variance of the number of points remaining in the game

Let $\sigma^2(Y_A^{pg}(a,b))$ and $\sigma^2(Y_B^{pg}(a,b))$ represent the variance of the number of points remaining in a game at point score (a,b) for player A and B serving respectively

Recurrence formulas:
$\sigma^2(Y_A{}^{pg}(a,b))= p_A\, \sigma^2(Y_A{}^{pg}(a+1,b))+q_A\, \sigma^2(Y_B{}^{pg}(a,b+1))+p_Aq_A(\mu(Y_A{}^{pg}(a+1,b))-\mu(Y_B{}^{pg}(a,b+1)))^2$
$\sigma^2(Y_B{}^{pg}(a,b))= p_B\, \sigma^2(Y_B{}^{pg}(a,b+1))+q_B\, \sigma^2(Y_A{}^{pg}(a+1,b))+p_Bq_B(\mu(Y_B{}^{pg}(a,b+1))-\mu(Y_A{}^{pg}(a+1,b)))^2$

Boundary values:
$\sigma^2(Y_A{}^{pg}(a,b))=\sigma^2(Y_B{}^{pg}(a,b))=0$, if a=21 and b ≤19 or b=21 and a ≤19 or (a,b) = (22,20), (23,21), (24,22), (25,23), (26,24), (27,25), (28,26), (29,27), (30,28), (20,22), (21,23), (22,24), (23,25), (24,26), (25,27), (26,28), (27,29), (28,30)
$\sigma^2(Y_A{}^{pg}(29,29))=\sigma^2(Y_B{}^{pg}(29,29))=0$

p_A	p_B	$\mu(Y_A{}^{pg}(0,0))$	$\sigma(Y_A{}^{pg}(0,0))$	$\mu(Y_B{}^{pg}(0,0))$	$\sigma(Y_B{}^{pg}(0,0))$
0.46	0.46	37.63	3.91	37.63	3.91
0.48	0.46	37.51	3.92	37.50	3.92
0.50	0.46	37.32	3.95	37.30	3.95
0.52	0.46	37.06	4.00	37.05	4.00
0.48	0.48	37.43	3.92	37.43	3.92
0.50	0.48	37.31	3.93	37.30	3.94
0.52	0.48	37.11	3.98	37.11	3.98
0.50	0.50	37.24	3.94	37.24	3.94
0.52	0.50	37.10	3.96	37.10	3.96
0.52	0.52	37.03	3.97	37.03	3.97

Table: Mean and standard deviation of the number of points played in a game for different values of p_A and p_B, and for both player A and player B serving first

Mean number of games remaining in the match

Let $\mu(Y_A{}^{gm}(c,d))$ and $\mu(Y_B{}^{gm}(c,d))$ represent the mean number of games remaining in a match at game score (c,d) for player A and player B serving respectively

Recurrence formulas:
$\mu(Y_A{}^{gm}(c,d))= 1+p_A{}^g\, \mu(Y_A{}^{gm}(c+1,d))+q_A{}^g\, \mu(Y_B{}^{gm}(c,d+1))$
$\mu(Y_B{}^{gm}(c,d))=1+q_B{}^g\, \mu(Y_A{}^{gm}(c+1,d))+p_B{}^g\, \mu(Y_B{}^{gm}(c,d+1))$

Boundary values:
$\mu(Y_A{}^{gm}(2,0))= \mu(Y_A{}^{gm}(0,2))= \mu(Y_B{}^{gm}(2,0))= \mu(Y_B{}^{gm}(0,2))=0$
$\mu(Y_A{}^{gm}(1,1))= \mu(Y_B{}^{gm}(1,1))=1$

Let Let $\mu(Y_A{}^{gm}(a,b:c,d))$ and $\mu(Y_B{}^{gm}(a,b:c,d))$ represent the mean number of games remaining in a match at point and game score (a,b:c,d) for player A and player B serving respectively

$\mu(Y_A{}^{gm}(a,b:c,d))=1+P_A{}^{pg}(a,b)\mu(Y_A{}^{gm}(c+1,d))+(1-P_A{}^{pg}(a,b))\mu(Y_B{}^{gm}(c,d+1))$
$\mu(Y_B{}^{gm}(a,b:c,d))=1+P_B{}^{pg}(a,b)\, \mu(Y_A{}^{gm}(c+1,d))+(1-P_B{}^{pg}(a,b))\mu(Y_B{}^{gm}(c,d+1))$

Variance of the number of games remaining in the match

Let Let $\sigma^2(Y_A^{gm}(c,d))$ and $\sigma^2(Y_B^{gm}(c,d))$ represent the variance of the number of games remaining in a match at game score (c,d) for player A and player B serving respectively

Recurrence formulas:
$\sigma^2(Y_A^{gm}(c,d))=p_A^g\,\sigma^2(Y_A^{gm}(c+1,d))+q_A^g\,\sigma^2(Y_B^{gm}(c,d+1))+p_A^g\,q_A^g(\mu(Y_A^{gm}(c+1,d))-\mu(Y_B^{gm}(c,d+1)))^2$
$\sigma^2(Y_B^{gm}(c,d))=q_B^g\,\sigma^2(Y_A^{gm}(c+1,d))+p_B^g\,\sigma^2(Y_B^{gm}(c,d+1))+q_B^g p_B^g(\mu(Y_A^{gm}(c+1,d))-\mu(Y_B^{gm}(c,d+1)))^2$

Boundary values:
$\sigma^2(Y_A^{gm}(2,0))=\sigma^2(Y_A^{gm}(0,2))=\sigma^2(Y_B^{gm}(2,0))=\sigma^2(Y_B^{gm}(0,2))=0$
$\sigma^2(Y_A^{gm}(1,1))=\sigma^2(Y_B^{gm}(1,1))=0$

Let $\sigma^2(Y_A^{gm}(a,b:c,d))$ and $\sigma^2(Y_B^{gm}(a,b:c,d))$ represent the variance of the number of games remaining in a match at point and game score (a,b:c,d) for player A and player B serving respectively

$\sigma^2(Y_A^{gm}(a,b:c,d))=P_A^{pg}(a,b)\,\sigma^2(Y_A^{gm}(c+1,d))+(1-P_A^{pg}(a,b))\,\sigma^2(Y_B^{gm}(c,d+1))+P_A^{pg}(a,b)\,(1-P_A^{pg}(a,b))$
$(\mu(Y_A^{gm}(c+1,d))-\mu(Y_B^{gm}(c,d+1)))^2$
$\sigma^2(Y_B^{gm}(a,b:c,d))=P_B^{pg}(a,b)\,\sigma^2(Y_A^{gm}(c+1,d))+(1-P_B^{pg}(a,b))\,\sigma^2(Y_B^{gm}(c,d+1))+P_B^{pg}(a,b)(1-P_B^{pg}(a,b))$
$(\mu(Y_A^{gm}(c+1,d))-\mu(Y_B^{gm}(c,d+1)))^2$

p_A	p_B	$\mu(Y_A^{gm}(0,0))$	$\sigma(Y_A^{gm}(0,0))$	$\mu(Y_B^{gm}(0,0))$	$\sigma(Y_B^{gm}(0,0))$
0.46	0.46	2.51	0.50	2.51	0.50
0.48	0.46	2.50	0.50	2.50	0.50
0.50	0.46	2.48	0.50	2.48	0.50
0.52	0.46	2.45	0.50	2.45	0.50
0.48	0.48	2.50	0.50	2.50	0.50
0.50	0.48	2.50	0.50	2.50	0.50
0.52	0.48	2.48	0.50	2.48	0.50
0.50	0.50	2.50	0.50	2.50	0.50
0.52	0.50	2.49	0.50	2.49	0.50
0.52	0.52	2.50	0.50	2.50	0.50

Table: Mean and standard deviation of the number of games played in a match for different values of p_A and p_B, and for both player A and player B serving first