Kelly strikes again

By Tristan Barnett

The Kelly criterion is well-established in risk-taking games such as blackjack and the stock market to maximize the long-term growth of the bank (Kelly 1956). Barnett (2010) applied the Kelly criterion to litigation to determine whether it is beneficial for a victim to file a lawsuit against the injurer given there are risks involved if unsuccessful in court. This methodology was extended in Barnett (2011) to obtain an arbitration value, where the amount awarded to the victim is less than expectation and shown to be 'fairer' when compared with the amount obtained using the Von Neumann and Morgenstern game theory framework (Von Neumann and Morgenstern 1944). By utilizing the Kelly criterion, this article establishes the Kelly Equilibrium in a two-person zero-sum game; such that each player has a total of two strategies (more formally known as a 2 x 2 zero-sum game). The Kelly Equilibrium could be used in determining an arbitration value as an alternative to the payout given by the Nash Equilibrium.

Consider the 2 x2 zero-sum game below (call it Game 1). The Nash Equilibrium (or Minimax Theorem in a two-person zero-sum game) gives mixed strategies of P1: 4/9A, 5/9B and P2: 2/3A, 1/3B. The value of the game is 1/9. This implies that by P1 playing Nash Equilibrium strategies they are guaranteed an expected positive payout of 1/9 regardless of the strategies used by P2. However, by P1 playing Nash Equilibrium strategies there is always a probability of P1 ending up with a negative payout on any single trial and of even greater impact is the possibility of P1 ending up with the maximum possible loss (MPL) of -1. It was shown in Barnett (2011) that the standard deviation and coefficients of skewness and kurtosis were reduced by P1 playing strategies that reduced the probability of the MPL loss occurring; and hence the following definition was given for risk-averse strategies in a 2 x 2 zero-sum game.

Definition 1: Consider a 2 x 2 zero-sum game where at least one of the payouts is positive and at least one of the payouts is negative. The value v of the game under the Minimax Theorem is either positive, negative or zero. Risk-averse strategies can be obtained when v is positive such that the expected payout for P1 is positive regardless of the strategies used by P2, and the probability of obtaining the MPL for P1 in the game is reduced when compared with the strategies under the Minimax Theorem. Risk-averse strategies for P2 when v is negative follow.

		P2	
		A	B
P1	A	2/3	-1
	B	-1/3	1

Game 1

Using Definition 1, the risk-averse solution for P1 is obtained as P1: 1/3<A<4/9, 5/9<B<2/3. These calculations were obtained by noting that P1 always obtains an expected positive payout regardless of the strategies used by P2.

Suppose the payouts for each player in Game 1 was to be determined by an outside arbitrator. One obvious method is simply to use the Nash Equilibrium. For Game 1, this would be 1/9 to P1. However, it would appear to be more of an incentive for P1 (the favourable player) to have the game determined by arbitration rather than play the game simultaneously, as they run the risk of being at a loss even though the expected outcome is positive. For example, from table 1 if both players are playing Nash Equilibrium strategies, there is a 0.148 probability of ending up with the MPL of −1 on any trial and a 0.519 probability of ending up with any loss on any trial. P1 can of course reduce the MPL by playing risk-averse strategies but as a consequence could reduce the expected amount if the other player changed strategies accordingly. This illustration suggests that the arbitration amount to the favourable player should be less than the expected amount as given by the Nash Equilibrium.

Outcome	Payout	Probability	Expected Payout
AA	2/3	4/9 x 2/3 = 0.296	2/3 x 8/27 = 0.198
AB	-1	4/9 x 1/3 = 0.148	-1 x 4/27 = -0.148
BA	-1/3	5/9 x 2/3 = 0.370	-1/3 x 10/27 = -0.123
BB	1	5/9 x 1/3 = 0.185	1 x 5/27 = 0.185
		1	**0.111**

Table 1: Probabilities and expected payouts for Game 1 with both players using strategies under the Minimax Theorem

Suppose Game 1 was played as a casino game over many trials and P1 (the gambler) had a finite bank. If P1 bet the same amount on each trial, then the expected profit on each trial would be 0.1111 for a unit bet. The expected profit on each trial differs when using the Kelly criterion since the player's bankroll changes each trial according to the wins and losses, and hence we will adopt an averaged expected profit notation. If P1 applied the Kelly criterion, then the averaged expected profit on each trial would be 0.1111 × 0.2217 = 0.0246. Given the Kelly criterion maximizes the long-term growth of the bank, this would appear to be a reasonable method in a favourable gambling context for determining an arbitration value and shows that the averaged expected amount of profit is less than the amount given by fixed betting on each trial. Based on this reasoning, an arbitration value could be determined by the averaged expected profit as given under the Kelly criterion. For Game 1, this value is given as 0.0246. This analysis leads to the Kelly Equilibrium in a 2 x 2 zero-sum game as given by Definition 2.

Definition 2: Consider a 2 x 2 zero-sum game where at least one of the payouts is positive, at least one of the payouts is negative and the value of the game v is positive. The Kelly Equilibrium can be obtained by v.b, where b is given by the Kelly betting fraction and P1 then uses risk-averse strategies such that the expected payout is v.b with P2 playing strategies under the Expected Value Principle.

For Game 1, the Kelly Equilibrium is given by the strategies of P1: A=0.3579, B=0.6421 and P2: A=1, B=0 with a payout of 0.0246. Note how these strategies using the Kelly Equilibrium are within the risk-averse region of P1: 1/3<A<4/9, 5/9<B<2/3. Table 2 summarizes the Nash and Kelly Equilibrium.

	Game 1
Nash Equilibrium	P1: A=0.4444, B=0.5556 P2: A=0.6667, B=0.3333 Payout: 0.1111
Kelly Equilibrium	P1: A=0.3579, B=0.6421 P2: A=1, B=0 Payout: 0.0246

Table 2: The Nash Equilibrium and Kelly Equilibrium for Game 1

BARNETT, T. (2010). Applying the Kelly Criterion to Lawsuits. Law, Probability & Risk, **9(2)**, 139–147.
BARNETT, T. (2011). Obtaining a Fair Arbitration Value. Law, Probability & Risk, **10(2)**, 123–131.
KELLY, J. L. (1956) A New Interpretation of Information Rate. The Bell System Technical Journal, July, 917–926.
VON NEUMANN, J. AND MORGENSTERN, O. (1944). Theory of Games and Economic Behavior, Wiley.