How Much to Bet on Video Poker

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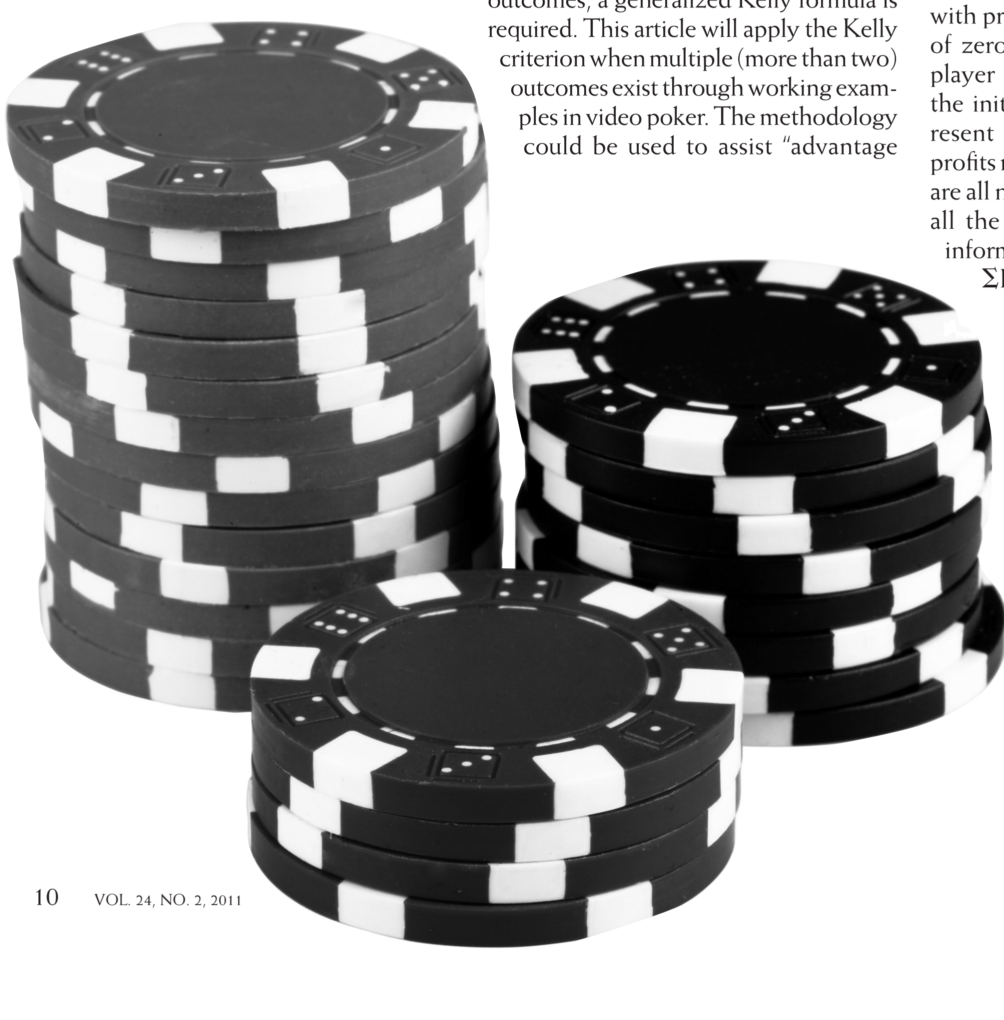
A question that arises whenever a game is favorable to the player is how much to wager on each event? While conservative play (or minimum bet) minimizes "large" fluctuations, it lacks the potential to maximize long-term growth of the bank. At the other extreme, aggressive play (or maximum bet) runs the risk of losing an entire bankroll, even though the player has an advantage in each trial of the game. What is required is a mathematical formulation that informs the player of how much to bet with the objective of maximizing the long-term growth of the bank.

The famous Kelly criterion, as developed by John L. Kelly in "A New Interpretation of Information Rate," achieves this objective. The Kelly criterion has been most recognized in games in which there are two outcomes: win \$x with probability p and lose \$y with probability 1-p. When there are more than two outcomes, a generalized Kelly formula is required. This article will apply the Kelly criterion when multiple (more than two) outcomes exist through working examples in video poker. The methodology could be used to assist "advantage players" in the decisionmaking process of how much to bet on each trial in video poker.



Kelly Criterion

Analysis of casino games and percent house margin

A casino game can be defined as follows: There is an initial cost C to play the game. Each trial results in an outcome O_i, where each outcome occurs with profit k_i and probability p_i. A profit of zero means the money paid to the player for a particular outcome equals the initial cost. Profits above zero represent a gain for the player; negative profits represent a loss. The probabilities are all non-negative and sum to one over all the possible outcomes. Given this information, the total expected profit $\Sigma E_i = \Sigma p_i k_i$. The percent house margin (%HM) is then $-\Sigma E_i/C$, and the total return is $1+\Sigma E_i/C$. Positive percent house margins indicate the gambling site, on average, makes money and the players lose money. Negative percent house margins indicate the game is favorable to the player and could possibly generate a long-term profit. Table 1 summarizes this information when there are m possible outcomes.

Classical Kelly criterion

The well-established classical Kelly criterion is given by the following result: Consider a game with two possible outcomes: win or lose. Suppose the player profits k units for every unit wager and the probabilities of a win and loss are given by p and q, respectively. Further, suppose that on each trial, the win probability p is constant with p + q = 1. If kp-q>0, so the game is advantageous to the player, then the optimal fraction of the current capital to be wagered is given by b*= (kp-q)/k.

Consider the following example. A player profits $2 with probability 0.35 and profits -$1 with probability 0.65, as represented by Table 2. Since the expected profit of 2 x 0.35-0.65=0.05>0, the game is advantageous to the player and the optimal fraction is given by b*= (2 x 0.35-0.65)/2 = 0.025. If a player has a $100 bankroll, then wagering 100*0.025 = $2.50 on the next hand will maximize the long-term growth of the bank. If the player loses $1 on that hand, then the next wager should be exactly 99*0.025 = $2.475 under the classical Kelly criterion. Since fractions are often not allowed in gambling games, this figure should be rounded down to an allowable betting amount.

Kelly criterion for multiple outcomes

When there are multiple (more than two) outcomes, as is the situation for video poker, a generalized Kelly formula is required from the classical Kelly formula. This generalized Kelly formula is given by Theorem 1 (see "A Proof of Theorem 1").

Theorem 1

Consider a game with m possible discrete finite mixed outcomes. Suppose the profit for a unit wager for outcome i is k_i with probability pi for $1 \le i \le m$, where at least one outcome is negative and at least one outcome is positive. Then, if $\sum_{i=1}^{m} k_i p_i > 0$ a winning strategy exists, and the maximum growth of the bank is attained when the proportion of the bank bet at each turn, b*, is the smallest positive root of

$$\sum_{i=1}^{m} \frac{k_i p_i}{1 + k_1 b^*} = 0$$

Let g(b) represent the rate of growth of the bank that is the quantity to be

A Proof of Theorem 1

Assume a constant proportion b of the bank is bet with m discrete finite mixed outcomes. Let $B(1)/B(0)$ equal $1+k_i b$ with probability p_i for i = 1 to m, where B(t) represents the player's bank at time t. Assume the player wishes to maximize g(b) = E[log(B(1) / B(0))] = . Without loss of generality, let k_1 be the maximum possible loss. In the interval $0 < b < -1/k_1$, $1+k_i b > 0$ since $k_i \ge k_1$ for i = 1 to m, so the logarithm of each term is real. Taking derivatives with respect to b,

$$\frac{dg(b)}{db} = \sum_{i=1}^{m} \frac{k_i p_i}{1 + k_1 b} = g'(b)$$

and

$$\frac{d^2 g(b)}{db^2} = \sum_{i=1}^{m} \frac{k_i^2 p_i}{(1 + k_1 b)^2} = g''(b)$$

Note that
(a) g(0) = 0
(b) g′(0) > 0 follows directly from the requirement for a winning strategy (so you should bet something)
(c) g′′(b) < 0 for $0<b<-1/k_1$ (where k_1 is the MPL) so the first derivative has at most one zero root in this interval.

Hence, whenever there is a winning strategy, the force of growth has a unique maximum given by the root of

$$\sum_{i=1}^{m} \frac{k_i p_i}{1 + k_1 b^*} = 0$$

Table 1—Representation in Terms of Expected Profit of a Casino Game with m Possible Outcomes

Outcome	Profit	Probability	Expected Profit
O_1	k_1	p_1	$E_1=p_1k_1$
O_2	k_2	p_2	$E_2=p_2k_2$
O_3	k_3	p_3	$E_3=p_3k_3$
…	…	…	…
O_m	k_m	p_m	$E_m=p_mk_m$
		1.0	ΣE_i

Table 2—A Sample Casino Game to Determine the Optimal Betting Fraction Under the Kelly Criterion

Outcome	Profit	Probability	Expected Profit
Win	$2	0.35	$0.70
Lose	-$1	0.65	-$0.65
		1.0	0.05

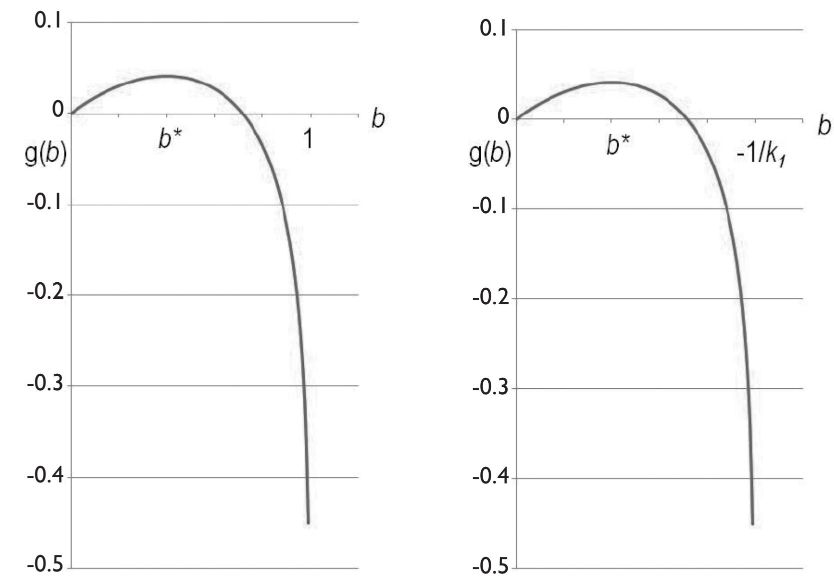


Figure 1. Graphical representation of the Kelly criterion for the classical case (left) and when multiple outcomes exist (right), where the optimal betting fraction of b* occurs at a maximum turning point on g(b). Value k_1 is the maximum possible loss in the multiple outcome game.

maximized. Figure 1 shows a graphical representation of the Kelly criterion for the classical case (left) and when multiple outcomes exist (right). Let the value k_1 be the maximum possible loss in the multiple outcome game. The player's bank will grow as long as g(b)>0 and is maximized when g′(b)=0 (represented by g(b*) in Figure 1). It is important to note that a player's bank will not grow (and is likely to hit ruin) when over betting the bankroll, even though the game is still favorable. This is represented on the graph for the values of b such that g(b)<0.

Kelly criterion for multiple outcomes: approximations to the optimal

According to Stanford Wong in his 1981 *Blackjack World* article, "Optimal bet size turns out to be the expected arithmetic win rate divided by the sum of the squares. For small expected win rates, such as you have in blackjack, the denominator is approximately equal to the variance."

This statement suggests approximating the optimal b* by

$$b' = \frac{\sum_{i-1}^{m} p_i k_i}{\sum_{i=1}^{m} p_i (k_i)^2}$$

The formula b′ is simple to compute given the probabilities and profits for a single play of a game.

According to WizardofOdds.com, "Most gamblers use advantage/variance as an approximation, which is a good estimator." This second approximation can be written as

$$b'' = \frac{\sum_{i=1}^{m} p_i k_i}{\sum_{i=1}^{m} p_i (k_i)^2 - \left(\sum_{i=1}^{m} p_i k_i\right)^2} = \frac{advantage}{variance}$$

Again, this formula is simple to compute given the probabilities and profits. It is obvious that 0<b′<b″ (the denominator of b′ is larger). It would be useful if we could prove b″<b*, because then b″ would be the superior approximation to b* always. Unfortunately, it is not so. A proof that it is not so is given in "A Demonstration That b″ Can Be Larger Than b*."

Despite this, there are available criteria to show when either approximation is useful for managing the risk of ruin from over betting. Observe in Figure 1 that g′(b) > 0 for 0 < b < b*. It is simple to check in practical examples if either

(a) $g'(b') > 0$ and $0 < b' < -1/k_1$ or

(b) $g'(b'') > 0$ and $0 < b'' < -1/k_1$

If condition (a) is satisfied, but condition (b) is not, then use b′. If condition (b) is satisfied, but condition (a) is not, then use b″. When both these sets of conditions are satisfied, it is preferable to work with b″, since it is a closer approximation to the optimal value b*. Notice the criteria do not require prior knowledge of the value of b*.

A Demonstration That b″ Can Be Larger Than b*

Suppose m=2 possible outcomes occur in a game. Suppose $k_1 + k_2 < 0$ with $k_1<0$ and $k_2>0$ and the conditions of Theorem 1 are satisfied. Then b*< b′, which implies that b*<b″ also, since b′<b″.

Proof:

Let $m_1 = p_1k_1 + p_2 k_2$ and $m_2 = p_1 k_1^2 + p_2 k_2^2$. Then

$$\begin{aligned}
b' - b^* &= m_1/m_2 + m_1/(k_1k_2) \\
&= m_1/(m_2 k_1 k_2)(m_2 + k_1 k_2) \\
&= m_1/(m_2 k_1 k_2)(p_1 k_1^2 + p_2k_2^2 + (p_1+p_2) k_1 k_2) \\
&= m_1/(m_2 k_1 k_2)(p_1 k_1 + p_2 k_2)(k_1+k_2) \\
&= m_1^2/(m_2 k_1 k_2)(k_1+k_2) \\
&>
\end{aligned}$$

0 since $k_1 < 0$ and $k_2 > 0$ and $k_1+k_2<0$.

Thus b′>b*.

Video Poker

Nonprogressive machines

Video poker is based on the traditional card game of draw poker. Each play of the video poker machine results in five cards being displayed on the screen from the number of cards in the pack used for that particular type of game (usually a standard 52-card pack, or 53 if the joker is included as a wild card). The player decides which of these cards to hold by pressing the hold button beneath the corresponding cards. The cards not held are randomly replaced by cards remaining in the pack. The final five cards are paid according to the payout table for that particular type of game. The pay tables follow the same order as traditional draw poker. For example, a full house pays more than a flush. Without a thorough understanding of video poker, it should be clear in the analysis to follow how Theorem 1 can be applied to determining an optimal bet size.

A pay table for the outcomes, profits, probabilities, and expected profits for a Jacks or Better machine (known as "All American Poker") are given in Table 3. The probabilities were obtained using WinPoker (a commercial product available at *www.zamzone.com*) and assume the player is always maximizing the expected profit on determining the correct playing strategies. Note that $1 is bet each game. It shows that the overall payback for this machine by playing an optimal strategy is 100.72%. The standard deviation is calculated as 5.18. The approximation formulas above give b′=0.0269436% and b″= 0.0269437%. Since g(b″)=0.000789 > 0 and 0 < b″< 1, either approximation of b′ and b″ is useful for managing the risk of ruin from over betting.

Theorem 1 is applied to determine a bet size for this video poker game by using the payouts and probabilities given in Table 3. The solver function in Excel is used to calculate this value as b*=0.030679%. Example: With a $10,000 bankroll, Theorem 1 suggests the player should initially bet $3.07 (likely to be round down to $3).

Progressive machines

Often, a group of machines is connected to a common jackpot pool, which continues to grow until someone gets a royal flush. When this occurs, the jackpot is reset to its minimum value.

Numerical Illustration of Kelly Criterion in Multiple (m>2) Outcome Game

Suppose m=3 and the outcome k_1=-1 occurs with probability 0.45, k_2=1 with 0.45, and k_3=2 with probability 0.10. The expected outcome is (-1)(0.45)+(1)(0.45)+2(0.10) = 0.20 >0, which is positive. The approximations are b′ = 0.2/[(-1)^2(0.45) + (1^2)*0.45 + (2^2)*0.10] = 0.2/1.3 = 0.1538 and b″ = 0.2/[1.3 – 0.2^2] = 0.2/1.26 = 0.1587. k_1 is -1 and both b′ and b″ are less than -1/k_1 = 1. g′(b′) = (-1)(0.45)/(1-1(b′)) + (1)(0.45)/(1+1(b′)) + 2(0.10)/(1+2(b′)) = 0.0111. Similarly, g′(b″) = 0.0053. Both conditions (a) and (b) are satisfied, so it is preferable to work with b″. If someone has $1,000, the bet should be $158.73, which likely would be rounded to $158.

Table 3—Profits and Probabilities for the All-American Poker Game

Outcome	Return ($)	Profit ($)	Probability	Expected Profit ($)
Royal Flush	800	799	1 in 43,450	0.018
Straight Flush	200	199	1 in 7,053	0.028
Four of a Kind	40	39	0.00225	0.088
Full House	8	7	0.01098	0.077
Flush	8	7	0.01572	0.110
Straight	8	7	0.01842	0.129
Three of Kind	3	2	0.06883	0.138
Two Pair	1	0	0.11960	0.000
Jacks or Better	1	0	0.18326	0.000
Nothing	0	-1	0.58076	-0.581
			1.00	**0.0072**

Table 4—Probabilities of Outcomes for Different Jackpot Levels for the All-American Poker Game

Outcome	Return ($)	Prob: $250 Jackpot	Prob: $800 Jackpot	Prob: $1,200 Jackpot
Royal Flush	Jackpot	1 in 58,685	1 in 43,450	1 in 35,848
Straight Flush	200	1 in 7,272	1 in 7,053	1 in 6,999
Four of a Kind	40	0.00226	0.00225	0.00225
Full House	8	0.01101	0.01098	0.01096
Flush	8	0.01588	0.01572	0.01505
Straight	8	0.01851	0.01842	0.01846
Three of Kind	3	0.06899	0.06883	0.06888
Two Pair	1	0.11988	0.11960	0.11954
Jacks or Better	1	0.18406	0.18326	0.18336
Nothing	0	0.57924	0.58076	0.58132
		1.00	**1.00**	**1.00**

Table 5—Kelly Criterion Analysis for Progressive Jackpot Machines

Jackpot	Return	Theorem 1	$11,000	$17,000
$250	99.62%	–	–	–
$800	100.72%	0.0307%	$3.38	$5.22
$1,200	101.74%	0.0468%	$5.15	$7.96

Usually, this minimum value would give a return of less than 100%, which creates a win-win situation for the astute player and the house. The amount bet to obtain the jackpot is a fixed amount. Table 4 represents the probabilities of outcomes with three jackpot levels for the "All-American Poker" game. The $800 jackpot was the game analyzed earlier. The $250 and $1,200 jackpots give returns of 99.62% and 101.74%, respectively. Notice the probability of obtaining a royal flush increases as the jackpot increases. This is logical, as a player would be more aggressive toward obtaining a royal flush with a larger jackpot.

Suppose a player has a bankroll of $11,000 and is required to bet $5 hands. What jackpot level is required to maximize the long-term growth of the player's bank under the Kelly criterion? Table 5 gives the results and can conclude that a jackpot level of $1,200 is required. A player would need a bankroll of about $17,000 to play the game at a jackpot level of $800.

Practical Difficulties

Despite the theoretical advances made above, it is impossible to effectively implement the optimal Kelly betting strategy on an All-American Poker machine, or any other video poker game. There are three main sources of difficulty. The first is the existence of a minimum betting unit on a machine. The second is the need to round the bet to avoid fractions of a unit. Third, to gain an edge in the long run requires hitting royal flushes. In the nonprogressive All-American Poker machine, this occurs on average once every 43,450 trials. Therefore, a player's bankroll would need to withstand the downward drift between hitting jackpots to avoid over betting.

Conclusions

An analysis of casino games was given to identity when games are favorable to the player and could possibly generate a long-term profit. Analyses were given for both the classical Kelly (two outcomes) and the Kelly criterion when multiple outcomes exist (more than two). The Kelly criterion when multiple outcomes exist was applied to favorable video poker machines. In the case of nonprogressive machines, an optimal betting fraction was obtained for maximizing the long-term growth of the player's bankroll. In the case of progressive machines, the minimum jackpot size was obtained as an entry trigger to avoid over betting, based on the player's bankroll. Approximation formulas when multiple outcomes exist were applied to video poker and shown to be useful for managing the risk. The analysis developed in this paper could be used by "advantage players" to assist with bankroll management, which is recognized as an important component to long-term success. **C**

Further Reading

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