Using Game Theory to Optimize Performance in a Best-of-N Set Match

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Introduction

In a sporting contest between two opposing sides, there are often strategic decisions to be made by the captain, coach or player, which could have an effect on the final outcome. Examples include when to take the batting power play in one-day cricket, when to pull the goalie in ice-hockey and how to optimize energy resources in a tennis match. A problem can be analyzed by assuming that one player/team makes all the decisions. More complex problems are where both players/teams are involved in the decision making process.

One area in which optimal allocation of energy is crucial is in contests with nested scoring systems (e.g. tennis). In these systems it is uncertain when the match will finish and players can win the match by winning well under half of the points played (Ferris 2003). Morris (1977), O'Donoghue (2001) and Pollard and Noble (2002) show that expending additional physical and mental effort on the important points in a game whilst relaxing on the unimportant points increases the chances of winning a game. In particular Morris (1977) states "*If he increased p from 0.60 to 0.61 on half his service points, and decreased from 0.60 to 0.59 on the unimportant half, he would increase his winning percentage by 0.0075 from 0.7357 to 0.7432*".

Other researchers have similarly addressed the question of optimal strategies to increase the probability of winning a tennis match. Barnett et al. (2004) concluded that an increase in effort is to be applied on every point of a game or every set of a match until either the game is finished or there are no increases remaining. Pollard and Pollard (2007) obtain optimal solutions for the maximum and minimum number of lifts available such that it is optimal not to lift and to lift respectively, for various point scores in a tiebreaker game and game scores in a tiebreaker set. Brimberg et al. (2004) model the situation where a player must allocate limited energy in a first-to-*n* match. They conclude that with only two possible energy choices for each game, it does not matter how energy is expended. With more than two possible energy choices, when the decision-maker falls behind in a match, s/he ought to switch to a more conservative approach by dividing her/his remaining energy evenly among all the possible remaining games.

The above references are modelled where only one player makes all the decisions in the match. This paper will account for the interactive nature of tennis and analyze a best-of-3 set match to include the situations where a) one player can apply an increase in effort on any set in the match, b) one player can vary effort about an overall mean, and where c) both players can apply an increase in effort on any set in the match. A conjecture is devised to obtain an optimal solution for a best-of-*N* set match, when both players can apply an increase in effort on any set in the match. The results from the analysis are used as a guide to strategy choices in sports where there are two opposing sides.

Analysis

Scenario a)

A best-of-3 set match is a contest where the first player to win 2 sets wins the match. Analyzing this system is non-trivial despite its relatively simple structure, because it is not certain that the third set will be played.

Consider the situation where a player has a constant probability p of winning a set. What is the probability of this player winning a best-of- 3 set match? The player can win the match by either winning in straight sets with probability p^2, losing the first set and winning the last two sets with probability $(1-p)p^2$ or winning the first and last set and losing the second set with probability $p(1-p)p$. Summing these, the probability of the player to win the match is given by $p^2(3-2p)$.

Now suppose a player increases his effort for one set at a match score in sets (e,f) (e= referred player's score, f=opponent's score), to change his probability of winning this set from p to $p+\varepsilon$, where $p+\varepsilon < 1$. This is equivalent to the opponent decreasing his effort at a match score (e,f) to change his probability of winning this set from $1-p$ to $1-p-\varepsilon$, since an increase of the probability of winning to one player is a decrease to the other player. On which set, should the player apply the increase to optimize their chances of winning the match? If the increase in effort is applied at (0,0), the probability for the player to win the match becomes $(p+\varepsilon)p(2-p) + (1-p-\varepsilon)p^2 = p^2(3-2p) + \varepsilon 2p(1-p)$. The same result is obtained if an increase in effort is applied at (1,1). Similarly the probabilities of a player to win the match when an increase in effort is applied at one of (1,0) or (0,1) is $p^2(3-2p) + \varepsilon p(1-p)$. Conditional on the match score reaching (1,0), the probability for a player to win the match when an increase in effort is applied at (1,0) or (1,1) is $p(2-p)+ \varepsilon(1-p)$; and conditional on the match score reaching (0,1), the chance for a player to win the match when an increase in effort is applied at (0,1) or (1,1) is

$p^2+ \varepsilon p$. Table 1 gives the increase in probability when effort is applied throughout the match. The first set played begins with the match score at (0,0). The third set is played only if the match score reaches (1,1). The second set played occurs with the match score at either (1,0) or (0,1). The probability of a player winning the match when one increase in effort is applied on the first, second or third set played is equal to $p^2(3\text{-}2p) + \varepsilon 2p(1\text{-}p)$. Using this result and the results represented in Table 1, an increase in effort could be applied on any set played within the match, and the player has optimized their chances of winning.

Scenario b)

Now suppose a player adopts a strategy of increasing his effort on the first, second or third set played by ε, and decreases p on the first, second or third set played (but a different set played from that of the increase) by ε, where $0 < p+\varepsilon < 1$. Calculations show the chance of the player winning the match for this situation is equal to $p^2 (3\text{-}2p)+ \varepsilon^2(2p\text{-}1)$. When p=½, $2p\text{-}1 = 0$, and there is no change in the chances for either player to win the match. When $p>$ ½, the chance for the player to win the match increases by $\varepsilon^2(2p\text{-}1)$ and therefore the opponent's chances to win the match decrease by $\varepsilon^2(2p\text{-}1)$. This implies that it is an advantage for the better player to vary his effort whilst maintaining his mean probability of winning a set. It follows by symmetry that the weaker player is disadvantaged by varying his effort.

Current match score	Match score at which an increase is applied	Increase in probability of winning match
(0,0)	(0,0)	$\varepsilon 2p(1\text{-}p)$
	(1,0)	$\varepsilon p(1\text{-}p)$
	(0,1)	$\varepsilon p(1\text{-}p)$
	(1,1)	$\varepsilon 2p(1\text{-}p)$
(1,0)	(1,0)	$\varepsilon(1\text{-}p)$
	(1,1)	$\varepsilon(1\text{-}p)$
(0,1)	(0,1)	εp
	(1,1)	εp
(1,1)	(1,1)	ε

Table 1. The increase in probability when effort is applied throughout the match.

Scenario c)

We now model the situation where both players can apply an increase in effort, which is represented by a two-person zero-sum game. For a best-of-3 set match, either player can apply an increase in effort at the first, second or third set played, resulting in a total of 9 possibilities. An increase in effort by ε at a set played from player A, results in increasing p to $p + \varepsilon$ ($p + \varepsilon < 1$), and an increase in effort by α at a set played from player B, results in decreasing p to $p - \alpha$ ($p - \alpha > 0$), where p represents the probability of player A winning a set. For the time being, it is assumed that both players must decide before the match has begun, on which set played that an increase is to be applied, and cannot change this choice throughout the match. Table 2 represents the probabilities of player A winning the match when an increase in effort is applied at the various sets played, where I_A and I_B represent an increase in effort at a set played by players A and B respectively. Notice that when both players apply an increase in effort on the same set played, the probability of player A winning the match is the same. Similarly, when both players apply an increase in effort on different sets played, the probability of player A winning the match is the same. When:

$p^2(3 - 2p) + 2p(1 - p)(\varepsilon - \alpha) + \alpha\varepsilon(2p - 1) > p^2(3-2p) + 2p(1 - p)(\varepsilon - \alpha)$

$\rightarrow \alpha\varepsilon(2p-1) > 0$

$\rightarrow p > ½$

Similarly when:

$p^2(3 - 2p) + 2p(1 - p)(\varepsilon - \alpha) + \alpha\varepsilon(2p - 1) < p^2(3-2p) + 2p(1 - p)(\varepsilon - \alpha)$

$\rightarrow \alpha\varepsilon(2p-1) < 0$

$\rightarrow p < ½$

I_A	I_B	Probability of player A winning
0	0	$p^2(3-2p) + 2p(1- p)(\varepsilon - \alpha)$
1	0	$p^2(3 - 2p) + 2p(1 - p)(\varepsilon - \alpha) + \alpha\varepsilon(2p - 1)$
0	1	$p^2(3 - 2p) + 2p(1 - p)(\varepsilon - \alpha) + \alpha\varepsilon(2p - 1)$
1	1	$p^2(3-2p) + 2p(1- p)(\varepsilon - \alpha)$
2	0	$p^2(3 - 2p) + 2p(1 - p)(\varepsilon - \alpha) + \alpha\varepsilon(2p - 1)$
0	2	$p^2(3 - 2p) + 2p(1 - p)(\varepsilon - \alpha) + \alpha\varepsilon(2p - 1)$
1	2	$p^2(3 - 2p) + 2p(1 - p)(\varepsilon - \alpha) + \alpha\varepsilon(2p - 1)$
2	1	$p^2(3 - 2p) + 2p(1 - p)(\varepsilon - \alpha) + \alpha\varepsilon(2p - 1)$
2	2	$p^2(3-2p) + 2p(1- p)(\varepsilon - \alpha)$

Table 2. Probability of player A winning the match when an increase in effort is applied by both players at a set played in a match.

The increase in probability of winning for the better player when an increase in effort for both players is applied on different sets, is a result of the variability about the overall mean, as presented in the above section. Let $X = p^2(3-2p) + 2p(1- p)(\varepsilon - \alpha)$and $Y = p^2(3 - 2p) + 2p(1 - p)(\varepsilon - \alpha) + \alpha\varepsilon(2p - 1)$. Let strategy

Ki (*K:* {A,B}, *i :* {1, 2, 3}) refer to player *K* applying an increase in effort at *i* sets played. The game theory matrix is represented by:

	B1	B2	B3
A1	X	Y	Y
A2	Y	X	Y
A3	Y	Y	X

This matrix can easily be solved and the results indicate that players A and B should apply mixed strategies of A: (⅓, ⅓,⅓) and B: (⅓, ⅓,⅓). The value (v) of the game is then ⅓X + ⅔Y. For example if $p = 0.25$, $\varepsilon = \alpha = 0.1$, then $X = 0.15625$; $Y = 0.15125$ and $v = 0.1529$.

Suppose either player can now alter their strategies as the match is in progress. When should either player apply an increase in effort to optimize the usage of their available increase?

Consider the following analysis. Suppose at the start of the match, player A decides to apply an increase in effort at the first set played with a score line of (0,0), and player B decides to apply an increase in effort at the third set played with a score line (1,1). After the first set has been played, player B now has a decision to make on whether to stay with the initial strategy, by applying an increase in effort at the third set, or change strategies and apply an increase in effort at the second set played. As previously calculated in the above section, player B has the same probability of winning the match by applying an increase at the second or thirds sets played. Therefore player B could change their initial strategy by applying an increase in effort at the second set, and have optimized the usage of their available increase. Similarly, if player B decides at the start of the match to apply an increase in effort at the first set played, and player A decides to apply an increase in effort at the third set played, then player A could change their initial strategy by applying an increase in effort at the second set, and have optimized the usage of their available increase. This analysis is summarized as follows:

1. Both players are to apply an increase in effort at the first set played with probability of ⅓ .
2. If one player applies an increase in effort at the first set played, then the other player can decide to apply an increase in effort at either the second or third sets played. If neither player increased their effort at the first set played, then both players are to apply an increase in effort at the second set played with probability of ½

3. If the match reaches (1,1) and neither player has applied their increase in effort, then the increase in effort by both players must be applied at this state of the match.

Note that the mixed strategies of A: (⅓, ⅓,⅓) and B: (⅓, ⅓,⅓) still gives an optimal solution.

A best-of-3 set match is extended to a best-of-N set match in the case N odd by the following conjecture:

Suppose both players can apply one increase in effort in a best-of-N set match (N: odd integer). An increase in effort by ε at a set played from player A, results in increasing p to $p + \varepsilon$ ($p + \varepsilon < 1$), and an increase in effort by α at a set played from player B, results in decreasing p to $p - \alpha$ ($p - \alpha > 0$), where p represents the probability of player A winning a set. Then an optimal strategy for both players is to decide at the start of the match to apply the increase in effort with equal probability at N sets played, where the probability of applying the increase in effort at a set is given by $1/N$. The value of the game is given by $1/N\ X + (N\text{-}1)/N\ Y$, where X represents the probability of player A winning the match when an increase in effort by each player is applied at the same set played, and Y represents the probability of player A winning the match when an increase in effort by each player is applied at different sets played.

Applications

It was shown from Scenario b), where one player can vary effort around an overall mean, that it is advantageous for the better player to vary his effort whilst maintaining his mean probability of winning a set. It follows that the weaker player is disadvantaged by varying his effort. This finding is in line with those of Mulvey et. al. (2002), who modelled the distribution of scores in golf from a selection of players. Their results showed that the lowest mean scores do not necessarily imply a greater chance of winning, since the standard deviation of players' scores also contribute to the chances of winning. This finding also has other applications to sports, as variability can enhance performance. For example, tactical decisions need to be made during a middle or long distance running event on whether an athlete should break away from the leading pack early or to stay with the leading pack in an attempt for a sprint finish. An athlete may be better off to break away from the leading pack early as this could be their best chance of finishing in a reasonable position even though they possibly run the risk of finishing in the bottom positions by utilizing energy resources too early. This example attempts to show that an athlete may be better off by increasing their variability by taking levels of risk, even though their overall mean performance may be better by not taking such levels of risk.

It was shown from Scenario c), where both players can apply an increase in effort,- that when both players apply an increase in effort on a particular set in a best-of-*N* set match, mixed strategies for both players should be applied where the probability for each set is equally likely. It was shown from Scenario a), where one player can apply an increase in effort on any set in the match,-that applying this increase in effort on any set played within the match, optimizes the player's chances of winning. This also implies that a player could optimize their chances of winning by applying an increase in effort on each set played with a probability, where the probabilities of the increase for each set are equally likely. Since a solution to Scenario a) is equivalent to the solution for Scenario c), then a player can be no worse off by analyzing Scenario c) - where both players/teams are optimizing their energy resources.

Mixed strategy solutions in game theory, such as those applied here, assume that the probabilities are randomized and this could have applications in other sports. For example, the batting power play in one-day cricket, when to pull the goalie in ice-hockey and when to interchange players in football could be randomized. Tennis players are known to use an occasional change-up tactic on serve as a surprise factor. This could involve a serve-and-volley or a slower paced first serve with heavy topspin. Randomizing this decision process could improve performance, based on the expectations of the opponent.

Conclusions

It has been shown that when only one player can apply an increase in effort on any set in a best-of-3 set match, their chances of winning the match are the same irrespective of which set the increase is applied. However, variability about an overall mean for a best-of-3 set match gave an increased probability of winning the match for the better player. By analyzing a best-of-3 set match, where each player can apply one increase in effort, a mixed strategy solution was obtained, where an optimal strategy for each player was given by A: (⅓, ⅓, ⅓) and B: (⅓, ⅓, ⅓). A conjecture is devised for a two-person zero-sum to obtain an optimal solution for a best-of-*N* set match. Some applications are given to the theoretical results, which could be used by coaches and players to optimize performance by increasing variability and randomising decision events.

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