A RECURSION METHOD FOR EVALUATING THE MOMENTS OF A NESTED SCORING SYSTEM

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Abstract. Nested scoring systems exist in several sports, e.g. tennis, badminton, table tennis, squash and volleyball. Recursive models are developed to study the influence that the scoring system has on the length of a match. Our examples are taken from the sport of tennis, which has rules covering points in a game, games in a set, and sets in a match. The assumption is made in the models that the transition probabilities from each state of the scoreboard depend only upon the score that has been reached, and is independent of the path by which that score was attained. This assumption enables us to show that, even in situations where the probability of a player winning a point on serve is not constant, there exists a simple calculus to determine higher moments of the number of points played at each level of the scoring system. A method is described for estimating the distribution of the total number of points in a match from its moments. This distribution is often multi-modal, and due care must be taken with its tails, as it is well known that a distribution is not uniquely determined by its moments.

Keywords: nested scoring systems, moment generating functions

INTRODUCTION

Nested scoring systems exist in several sports, e.g. tennis, badminton, table tennis, squash and volleyball. We are particularly interested in modelling situations where the probability of a player winning a point on serve is not constant.

Recursive models to evaluate the probability of winning a match under the various scoring systems are easy to develop provided sufficiently strong assumptions are made. Calculation with these models is straightforward when the assumption is made that the probability of the server winning a point is constant, and point outcomes are independent. However when attention is turned to the influence that a scoring system has on the length of a match, the analysis using recursive models becomes more difficult, especially when higher moments of the distributions are required. Monte Carlo methods have been used in the past, e.g. Pollard and Noble (2002, 2003, 2004) to consider cases in which the probability of winning a point is constant, and cases in which it is not constant. Our aim here is to show that methods can be developed to analyse these models using formulas that are both exact and fast to compute, and thus provide an alternative to simulation methods.

STOCHASTIC CALCULUS FOR MARKOV PROCESSES ON A LATTICE

Modelling processes on a graph

In tennis the match progresses as a player (or team) either wins or loses a point at each serve. The scoreboard accumulates the points won by each player, and at certain stages, depending on the rules, the scores accumulate at a higher level, and scores at the lower level are reset to zero. The rules may also impose various conditions on the rotation of serve between the players. Tennis has three higher levels known as game, set and match. We are interested as to how these levels nest within each other, and the construction the graph(s) for each level.

Modelling processes on a chain

Suppose the various states of the model form a chain. Denote the j^{th} state by E_j, and suppose the probability of being in the j^{th} state is $p\{E_j\} = p_j$.

Figure 1. Links in a chain

Consider the link in the chain from E_j to E_k, and denote the transition probability from state j to state k by

$$p\{(E_j, E_k)\} = q_{j,k}$$

The conditional probability of reaching state k after being in state j is then

$$p\{E_k \mid E_j\} = p_j\, q_{j,k}$$

Let $X_{j,k}$ be a random variable that is associated with the link in the chain from E_j to E_k. Denote its moment generating function by $M_{Xj,k}(t)$, where $M_{Xj,k}(t) = E[\exp(t\, X_{j,k})]$

We assume that all its moments exist. Let $S_k = S_j + X_{j,k}$ and assume S_j and $X_{j,k}$ are independent. Then using a standard lemma on the sum of independent random variables it is easy to obtain

$$M_{Sk}(t) = M_{Sj}(t)\, M_{Xj,k}(t)$$

We can combine probabilities and moment generating functions to define the conditional moment generating function (Cmgf) on a chain.

Cmgf for the transition step.

$$T_{Xj,k}(t) = q_{j,k}\;\; M_{Xj,k}(t)$$

This is obtained by combining the multiplicative factors for the step of the chain.

Cmgf for the conditional outcome $\{E_k \mid E_j\}$ on a chain.

$$C_{Sk}(t) = T_{Xj,k}(t)\, C_{Sj}(t)$$

This relation is a multiplicative form, and can be applied on the chain at each successive link.

$$C_{Sk}(t) = C_{S0}(t)\, T_{X0,1}(t)\, T_{X1,2}(t) \ldots\ldots\ldots\, T_{Xj,k}(t)$$

To initialise this relation we consider the initial state of the chain, E_0. If we set

$p_0 = 1$ and $S_0 = 0$, so that $M_{S0}(t) = 1$, which leads to $C_{S0}(t) = 1$.

Modelling processes on a lattice

A lattice is a generalization of a chain where there may be joins as well as branches at various nodes as in Figure 2.

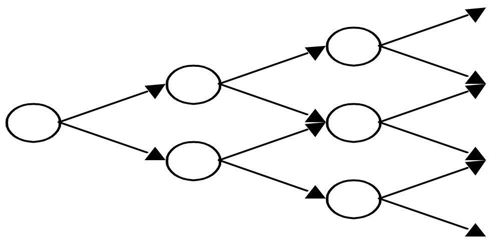

Figure 2. A lattice with 2-way branching and 2-way joining

It is straightforward to treat a branch exactly the same as a link on a chain. However the treatment at a join requires further consideration. We assume throughout that models of processes on a lattice have the following

A 1. Markov property: The transition probabilities from each state depend only upon that state, and not on the path by which the state was attained.

Suppose that there are n links that join at the k^{th} node. Since the links are disjoint, the probability of reaching state k, when the Markov property exists, is the sum of the transition probabilities weighted by the probability of being in the prior states.

$$p_k = p\{E_k\} = \sum_{j=1}^{n} p_j \, p\{E_k \mid E_j\} = \sum_{j=1}^{n} p_j \, q_{j,k}$$

Now considering the link from the j^{th} node, where

$M_{Sk}(t) = M_{Xj,k}(t)\, M_{Sj}(t)$ with probability $w_{j,k} = p_j\, q_{j,k} / p_k$, the probability weights satisfy

$$\sum_{j=1}^{n} w_{j,k} = \sum_{j=1}^{n} p_j\, q_{j,k} / p_k = 1,$$

so it is quite easy to calculate the moment generating function $M_{Sk}(t)$ as an expectation

$$M_{Sk}(t) = \sum_{j=1}^{n} w_{j,k}\, M_{Xj,k}(t)\, M_{Sj}(t) = \sum_{j=1}^{n} p_j\, q_{j,k}\, M_{Xj,k}(t)\, M_{Sj}(t) / p_k \,,$$

which leads to

$$C_{Sk}(t) = \sum_{j=1}^{n} T_{Xj,k}(t)\, C_{Sj}(t)$$

These calculations for lattice models with the Markov property can be summarised by the following two rules:

(a) for each branch, multiply by the Cmgf of the transition to progress on each link,

(b) for each join, multiply by the Cmgf of the transition to progress on each link and add.

The flexibility of these lattice models and the simplicity of their calculus enable us to study a wide range of interesting cases. However the rules of tennis allow a countably infinite number of points to be played in a game. Hence we require additional assumptions before the lattice model is tractable. A sufficient assumption at the game level is the following:

A2. The probability of the server winning a point depends only upon the margin in points between the two players, and does not depend on the total number of points played.

A similar problem arises in an advantage set. A sufficient assumption at the set level is:

A3. The probability of the server winning a game depends only upon the margin in games between the two players, and does not depend on the total number of games played.

The two assumptions A2 and A3 are sufficient to restrict to a finite number the distinct points that must be modelled.

Often an additional assumption is added at the match level:

A4. The probability of the server winning a set depends only upon the margin in sets between the two players, and does not depend on the total number of sets played.

Assumption A4 is appended to the other assumptions purely for computational convenience.

APPLICATIONS OF THE STOCHASTIC CALCULUS TO TENNIS

A tennis match

The example given here is for a tennis match consisting of best-of-3 advantage sets. Many other variations are possible; nevertheless this example is sufficient to illustrate the ease with which nested levels of scoring can be handled in tennis, even when the probability of the players winning a point on serve is not constant.

Denote the two players by A and B. Let p_A and q_A be the probability of a server A winning and losing a point on serve respectively, where $p_A + q_A = 1$.

Let p_B and q_B be the corresponding symbols for server B. We will ignore stating this correspondence for the many other symbols that we introduce. To ensure that we have a Markov process, we assume throughout that both p_A and p_B depend only upon the scoreboard at the time, and are independent of the path by which that score was attained. We do not assume that p_A and p_B are constant, except where explicitly stated. We note however that our numerical tables are restricted to cases where p_A and p_B are constant, for brevity.

Modelling a point on serve

Assume player A is the server for a point. The scoreboard takes a two-way branch when a point has been played. The moment generating function for a single point is e^t. The Cmgf for a winning step in the game when the point scores are (a,b) is

$T(w,t|g,A,a,b) = p_A e^t$ and the Cmgf for a losing step is $T(l,t|g,A,a,b) = q_A e^t$

where p_A and q_A may themselves depend upon the scoreboard. In this notation, g indicates we are working at the game level, A indicates the server, whilst a and b indicate the respective point scores before the current point is played.

A classical or advantage game

Assume player A is the server in a game. Denote the probability that the point score in a game for server A reaches (a, b) by P(a,b|g,A). Denote the conditional moment generating function for the number of points played when reaching this score by C(a,b,t|g,A). Note that 0,1,2,3 correspond to love, 15,30, 40,.

The initial value is $C(0,0,t|g,A) = 1$,

and the forward recursions are given by

$C(a,b,t|g,A) = T(w,t|g,A,a-1,b)\ C(a-1,b,t|g,A) + T(l, t|g,A,a,b-1)\ C(a,b-1,t|g,A)$ for $0 < a \leq 3$, $0 < b \leq 3$,
$C(a,b,t|g,A) = T(w,t|g,A,a-1,b)\ C(a-1,b,t|g,A)$ for $b = 0$, $0 < a \leq 3$ and for $a = 4$, $0 \leq b \leq 2$,
$C(a,b,t|g,A) = T(l,t|g,A,a,b-1)\ C(a,b-1,t|g,A)$ for $a = 0$, $0 < b \leq 3$ and for $b = 4$, $0 \leq a \leq 2$.

Player A wins the game when the point score reaches $a = 4$, $0 \leq b \leq 2$, and player B wins the game when the point score reaches $b = 4$, $0 \leq a \leq 2$. However when the score reaches (3, 3), also known as deuce, the game continues until one player is two points ahead, and this player wins the game. This rule permits the game to go on for an indefinite number of points. The forward recursions after deuce are given, for $n \geq 3$ by

$C(n+1,n,t|g,A) = T(w,t|g,A,n,n)\ C(n,n,t|g,A)$
$C(n,n+1,t|g,A) = T(l,t|g,A,n,n)\ C(n,n,t|g,A)$
$C(n+2,n,t|g,A) = T(w,t|g,A,n+1,n)\ C(n+1,n,t|g,A)$
$C(n,n+2,t|g,A) = T(l,t|g,A,n,n+1)\ C(n,n+1,t|g,A)$
$C(n+1,n+1,t|g,A) = T(w,t|g,A,n,n+1)\ C(n,n+1,t|g,A) + T(l,t|g,A,n+1,n)\ C(n+1,n,t|g,A)$

Let c and d denote the respective scores in games at the start of a game. The join of the instances where player A wins the game whilst serving is just one of four possible cases; the other cases are A losing whilst serving, B winning whilst serving and B losing whilst serving. The Cmgf for the number of points played when player A wins the game whilst serving is

$$T(w,t|s,A,c,d) = C(4,0,t|g,A) + C(4,1,t|g,A) + C(4,2,t|g,A) + C(5,3,t|g,A) \ldots .$$

Two points are played between consecutive deuces. It follows that, for $n \geq 3$,

$C(n+1,n+1,t|g,A) = R(t|g,A)\ C(n,n,t|g,A)$

where $R(t|g,A) = T(w,t|g,A,n,n+1)\ T(l,t|g,A,n,n) + T(l,t|g,A,n+1,n)\ T(w,t|g,A,n,n)$

Assumptions A1 and A2 are sufficient to ensure that R(t|g,A) is does not depend on n. Likewise 2 points are played to win after a deuce, so for $n \geq 3$: $C(n+2,n,t|g,A) = T(w,t|g,A,n+1,n)\ T(w,t|g,A,n,n)\ C(n,n,t|g,A)$.

These relations are used to obtain a closed form of the Cmgf for the transition step in a set as

$T(w,t|s,A,c,d)$

$= C(4,0,t|g,A) + C(4,1,t|g,A) + C(4,2,t|g,A) + T(w,t|g,A,4,3)\ T(w,t|g,A,3,3)\ C(3,3,t|g,A) /(1 - R(t|g,A))$

This closed form deals with convergence, and greatly assists the speed of computation. In the special case where p_A is constant it is easy to recover standard results such as

$$T(w,t|s,A,c,d) = p_A^4 e^{4t} + 4\ p_A^4 q_A e^{5t} + 10\ p_A^4 q_A^2 e^{6t} + 20\ p_A^5 q_A^3 e^{8t} / (1 - 2p_A q_A e^{2t})$$

when $0 < p_A < 1$. In general, where the closed form is not so tractable, numerical calculations using the recursive relations are usually carried through up to the fourth moment.

A serve win	A serve lose	B serve lose	B serve win	all games
0.8296	0.1704	0.2643	0.7357	1.0000
5.0409	1.1473	1.8002	4.6840	6.3362
17.5823	4.3858	7.0574	17.3191	23.1723

Table 1. Cmgfs for a game, showing first three coefficients, when $p_A = 0.65$, $p_B = 0.60$.

The advantage set

In an advantage set the serve alternates each successive game. Denote the Cmgf for the number of points played in a winning game for player A by T(w,t|s,A,c,d) when A is serving, and by T(l,t|s,B,c,d) when B is serving. Denote the probability that the game score in a set with A serving first reaches (c, d) by P(c,d|s,A). Denote the conditional moment generating function for the number of points played when reaching this score by C(c,d,t|s,A).

Assume player A is the server *in the first game* of the set. The initial value is C(0,0,t|s,A) = 1.

When $c+d = 1 \bmod 2$, player A has just served, so the forward recursions are given by

C(c,d,t|s,A) = T(w,t|s,A,c-1,d) C(c-1,d,t|s,A) + T(l,t|s,A,c,d-1) C(c,d-1,t|s,A) for $0 < c \leq 5$, $0 < d \leq 5$

C(c,d,t|s,A) = T(w,t|s,A,c-1,d) C(c-1,d,t|s,A) for $d = 0$, $0 < c \leq 5$ and for $c = 6$, $0 \leq d \leq 3$, and

C(c,d,t|s,A) = T(l,t|s,A,c,d-1) C(c,d-1,t|s,A) for $c = 0$, $0 < d \leq 5$ and for $d = 6$, $0 \leq c \leq 3$.

When $c+d = 0 \bmod 2$, player B has just served, so the forward recursions are now given by

C(c,d,t|s,A) = T(l,t|s,B,c-1,d) C(c-1,d,t|s,A) + T(w, t|s,B, c,d-1) C(c,d-1,t|s,A) for $0 < c \leq 5$, $0 < d \leq 5$

C(c,d,t|s,A) = T(l,t|s,B,c-1,d) C(c-1,d,t|s,A) for $d = 0$, $0 < c \leq 4$ and for $c = 6$, $0 \leq d \leq 4$, and

C(c,d,t|s,A) = T(w,t|s,B,c,d-1) C(c,d-1,t|s,A)for $c = 0$, $0 < d \leq 4$ and for $d = 6$, $0 \leq c \leq 4$.

A player wins the set when his game score is two ahead after at least six games have been played. This rule permits the advantage set to go on for an indefinite number of games. When the score is level at (5, 5), the set continues until one player is two games ahead, and this player wins the set. If player A serves first, the forward recursions after the scores are level are given, for $n \geq 5$, by

C(n+1,n,t|s,A) = T(w,t|s,A,n,n) C(n,n,t|s,A),

C(n,n+1,t|s,A) = T(l,t|s,A,n,n) C(n,n,t|s,A),

C(n+2,n,t|s,A) = T(l,t|s,B,n+1,n) C(n+1,n,t|s,A),

C(n,n+2,t|s,A) = T(w,t|s,B,n,n+1) C(n,n+1,t|s,A), and

C(n+1,n+1,t|s,A) = T(l,t|s,B,n,n+1) C(n,n+1,t|s,A) + T(w,t|s,B,n+1,n) C(n+1,n,t|s,A).

Let e and f denote the respective scores in sets. At the match level there are eight cases to consider. Let v and o denote whether an even and odd number of games is played in the set respectively. The Cmgf for the number of points in a set when A is serving first, and A wins in an odd number of games, is given by

T(w,o,t|m,A,e,f) = C(6,1,t|s,A) + C(6,3,t|s,A)

The Cmgf for the number of points in a set when A is serving first, and A wins in an even number of games, is given by T(w,v,t|m,A,e,f) = C(6,0,t|s,A) + C(6,2,t|s,A) + C(6,4,t|s,A) + C(7,5,t|s,A) +… .

Since two games are played between consecutive level scores in a set it follows that

C(n +1,n + 1, t|s,A) = R(t|s,A) C(n,n, t|s,A) for $n \geq 5$,

where R(t|s,A) = T(w,t|s,A,n,n) T(w,t|s,B,n+1,n) + T(l,t|s,A,n,n) T(l,t|s,B,n,n+1).

It is easy to check that R(0|s,A) $\neq 1$ provided $0 < p_A < 1$ and $0 < p_B < 1$. Using

C(n +2,n, t|s,A) = T(w,t|s,A,n,n) T(l,t|s,B,n+1,n) C(n,n, t|s,A) for $n \geq 5$,

we obtain a closed form for a transition step in a match as

T(w,v,t|m,A,e,f) = C(6,0,t|s,A) + C(6,2,t|s,A) + C(6,4,t|s,A)

+ C(5,5,t|s,A) T(w,t|s,A,5,5) T(l,t|s,B,6,5)/(1 - R(t|s,A)).

The two cases where A serves first in a set but loses the set follow a similar pattern. There are another four cases in a set where B serves first, but we omit the details.

A sv fst win/ A next	A sv fst win/ B next	A sv fst lose/ A next	A sv fst lose/ B next
0.3513	0.3226	0.2922	0.0339
27.8935	17.3168	21.8576	1.8884
1287.6826	477.6359	929.6579	53.8005

Table 2. Some Cmgfs for a set, showing first three coefficients, when p_A= 0.65, p_B= 0.60.

The best-of-3 sets match

A rotation of serve occurs between the two players for successive games throughout the match. Denote the probability that the set score in a match reaches (e, f) with A to serve in the next set by P(e,f,A|m). Denote the conditional moment generating function for the number of points played when reaching this score with A to serve by C(e,f,A,t|m).

The two initial values are C(0,0,A,t|m) = 1, and C(0,0,B,t|m) = 0.

and the forward recursions are given by

$$C(e,f,A,t|m) = T(w,v,t|s,A,e-1,f)\ C(e-1,f,A,t|m) + T(l,o,t|s,B,e-1,f)\ C(e-1,f,B,t|m)$$

for e = 1 or 2, f = 0 and e = 2, f = 1,

$$C(e,f,A,t|m) = T(l,v,t|s,A,e,f-1)\ C(e,f-1,A,t|m) + T(w,o,t|s,B,e,f-1)\ C(e,f-1,B,t|m)$$

for e = 0, f = 1 and e = 1, f = 2,

$$C(e,f,B,t|m) = T(w,o,t|s,A,e-1,f)\ C(e-1,f,A,t|m) + T(l,v,t|s,B,e-1,f)\ C(e-1,f,B,t|m)$$

for e=1, f = 0 and e = 2, f = 1, and

$$C(e,f,B,t|m) = T(l,o,t|s,A,e,f-1)\ C(e,f-1,A,t|m) + T(w,v,t|s,B,e,f-1)\ C(e,f-1,B,t|m)$$

for e = 0, f = 1 or 2 and e = 1, f = 2.

Otherwise for e = 1, f = 1 we have the pair of recursions which encompass 4-way joins.

$$C(e,f,A,t|m) = T(w,v,t|s,A,e-1,f)\ C(e-1,f,A,t|m) + T(l,o,t|s,B,e-1,f)\ C(e-1,f,B,t|m) + T(l,v,t|s,A,e,f-1)\ C(e,f-1,A,t|m) + T(w,o,t|s,B,e,f-1)\ C(e,f-1,B,t|m),$$

and

$$C(e,f,B,t|m) = T(w,o,t|s,A,e-1,f)\ C(e-1,f,A,t|m) + T(l,v,t|s,B,e-1,f)\ C(e-1,f,B,t|m) + T(l,o,t|s,A,e,f-1)\ C(e,f-1,A,t|m) + T(w,v,t|s,B,e,f-1)\ C(e,f-1,B,t|m).$$

We can study the case where player B serves in the first game of the match by reversing the pair of initial values, or the effect of randomizing the toss by setting both initial values to 0.5.

2 set match A sv fst win	2 set match A sv fst lose	3 set match A sv fst win	3 set match A sv fst lose
0.4402	0.1132	0.3009	0.1456
55.4046	15.3568	59.6526	29.3918
3595.7443	1067.3448	6096.1103	3057.9805

Table 3. Some Cmgfs for a match, showing first three coefficients, when p_A= 0.65, p_B= 0.60.

RECOVERING DISTRIBUTIONS FROM MOMENTS

We have shown above that using lattice models with the Markov property and a few other modest assumptions it is feasible to calculate moments of the number of points in a match.

We can write $C(w,t|m,A,0,0) = P(w|m,A,0,0)\ M_{Sw|m,A}(t)$ which enables us to recover both the probability of a player A winning, and the moment generating function for the total number of points played in such a match. This may be put to practical use (Barnett and Pollard,2006). The estimation of the distribution of the total number of points in a best-of-3 sets match requires some care, since it may be bi-modal. It is well known that a distribution is not uniquely determined by its moments. To overcome these difficulties, the distributions for matches requiring 2 sets and those requiring 3 sets to complete were estimated separately. This can be done using the first four moments of each distribution and the Normal Power approximation (Pesonen, 1975). It is inappropriate to use this approximation in conjunction with the statistics for "All matches" as reported in the Table 4. Further examples of estimating the distribution can be found in Pollard, Barnett, Brown & Pollard, (2007).

Type of match	2 set	3 set	All matches
Probability	55.35%	44.65%	100.00%
Mean no of points	128.54	199.43	160.20
St dev no of points	22.13	35.16	45.44

Table 4. Some statistics for a match, when $p_A = 0.65$, $p_B = 0.60$.

APPLICATIONS

The rules for scoring systems differ both within and between various sports. We have taken examples from tennis, which is a sport with a rich set of rules covering points in a game, games in a set, and sets in a match. This variety should be sufficient to demonstrate the generality of our methodology. This stochastic calculus using conditional moment generating functions has a wide range of applications to other sports e.g. tennis, badminton, table tennis, squash and volleyball. The common feature is that the event space includes the scoreboard for each player (or team) and the server. The rules for incrementing the scoreboard and rotating the server vary both within and between sports.

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